On a generalization of the Siegel function to higher dimensions

Mémoire de M2 by Mai Tien Long

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0.1 Introduction

Let $L = \mathbb{Z} + \mathbb{Z}\tau$ be a lattice in $\mathbb{C}$ and consider the Siegel function

$$g(z,\tau) = e^{\frac{-1}{2}z\eta(z)}\sigma(z)\Delta(\tau)^{\frac{1}{12}}, \qquad z \in \mathbb{C}, \tau \in H^*$$

where $\eta(z)$ is the quasi-period map, σ is the Weirstrass sigma function, and $\Delta^{\frac{1}{12}}(\tau)$ is the square of the Dedekind-eta function. Explicitly, it is given by a product expansion

$$g_a(\tau) = -q_\tau^{(1/2)B_2(a_1)}e^{2\pi i a_2(a_1-1)/2}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n/q_z)$$

where $q_\tau = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$, $z = a_1 + a_2\tau$, $a_1, a_2 \in \mathbb{C}$, $B_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial.

If we let

$$\phi(z) = -\log|g(z,\tau)|$$

then $\phi(z) : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \setminus \{0\} \to \mathbb{R}$ does not depend on the choice of τ. It is a function on the torus, "transcendental", and smooth everywhere except at 0. The function ϕ has several properties:

- **Distribution relation**. For any $n \geq 2$ and $z \neq 0$,

$$\sum_{w\in\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau),nw=z}\phi(w) = \phi(z)$$

- **Kronecker second limit formula**. Consider the Eisenstein series

$$E_{u,v}(\tau,s) = \sum_{(m,n)\neq(0,0)} e^{2\pi i(mu+nv)}\frac{y^s}{|m\tau+n|^{2s}}$$

  where $\tau = x + iy$ is in the upper half plane and u, v are not both integer. This series converges absolutely for $Re(s) > 1$, and can be continued analytically to an entire function of s. Its value at $s = 1$ is given by

$$E_{u,v}(\tau,1) = -2\pi\log|g_{-v,u}(\tau)|$$

  where $g_{u,v}(\tau)$ is the Siegel function.

- **Giving rise to elliptic units**. Let K be an imaginary quadratic field and O_K its ring of integers. Let $\mathfrak{f} = (N) \subseteq O_K$ be a nontrivial ideal and $Cl(\mathfrak{f}) = I(\mathfrak{f})/P_{K,1}(\mathfrak{f})$ be its ray class group. By class field theory, there exists a unique abelian extension $K_\mathfrak{f}$ of K, called the ray class field modulo a conductor $\mathfrak{f}$, with

$$\sigma : Cl(\mathfrak{f}) \cong Gal(K_\mathfrak{f}/K)$$

where σ is the Artin reciprocity map. For any $C \in Cl(\mathfrak{f})$, take $\mathfrak{c} \in C$ a representative integral ideal. Then $\mathfrak{f}\mathfrak{c}^{-1} \subseteq K$ is a lattice in $\mathbb{C}$. The **Siegel-Ramachandra invariant** is defined as

$$\mathfrak{g}_{\mathfrak{f}}(C) = g^{12N}(1, \mathfrak{f}\mathfrak{c}^{-1})$$

Siegel and Ramachandra proved that the value $g_{\mathfrak{f}}(C)$ depends only on the class $C \in Cl(\mathfrak{f})$ and is contained in $K_{\mathfrak{f}}$. Moreover, if N has at least two prime factors, then $g_{\mathfrak{f}}(C) \in O^*_{K_{\mathfrak{f}}}$. Thus one can construct units in the ray class fields of imaginary quadratic fields using N- torsion points of certain elliptic curves using Siegel functions. In particular, on those elliptic curves,

$$\text{the term } e^{12N\phi(z)} \text{ is an algebraic unit.}$$

Naturally, one asks for an analogue of the function ϕ in higher dimensions, and hopes to construct algebraic units from its values at certain torsion points of abelian varieties. An interpretation to these questions is given in the article *On a canonical class of Green currents for the unit sections of abelian schemes* by Vincent Maillot and Damian Rössler. In particular, they consider an abelian scheme A of relative dimension g over S. Using Arakelov theory, they are able to construct a canonical class of real currents $\mathfrak{g}_A$ of type $(g-1, g-1)$ on the complex dual abelian scheme $A^\vee(\mathbb{C})$. When $g = 1$ and $S = Spec(O_K)$, this class of currents restricts to functions ϕ_σ on the torus $E_\sigma(\mathbb{C})$ for each embedding $\sigma : K \to \mathbb{C}$, and is equal to $2 \times$ the (Siegel) function $\phi(z)$. Moreover, they showed that the class of currents $\mathfrak{g}_A$ satisfies several properties:

- For any $n \geq 2$, $[n]_*\mathfrak{g}_A = \mathfrak{g}_A$. This property is a generalization of the distribution relation of the function ϕ that we have seen.
- When restricted to the complement of the zero section, the class of currents $\mathfrak{g}_A$ can be given by the $(g-1)$ part of the Bismut-Köhler analytic torsion form of the Poincaré bundle along the fibration $A(\mathbb{C}) \times_{S(\mathbb{C})} (A^\vee(\mathbb{C}) \setminus S_0^\vee(\mathbb{C})) \to A^\vee(\mathbb{C}) \setminus S_0^\vee(\mathbb{C})$, where $S_0^\vee$ is the image of the zero section of $A^\vee/S$. When $g = 1$, the function ϕ_σ is given by the Ray-Singer analytic torsions of flat line bundles on the torus $E_\sigma(\mathbb{C})$. By an explicit calculation of Ray and Singer [21], this function is exactly $\frac{1}{\pi}\times$ the Eisenstein series in the Kronecker's second limit formula.
- After multiplying with some integer number, the pull-back of the Bismut-Köhler analytic torsion form of the Poincaré bundle along a non-trivial torsion section is contained in the image of the Beilinson regulator map from Quillen's algebraic K_1 group of S. When $S = Spec(O_K)$, $K_1(Spec(O_K)) = O_K^*$, and the Beilinson regulator map becomes the Dirichlet's regulator map, given by logarithm functions. This gives a link to the construction of algebraic units using Siegel functions. In particular, the property implies that when $g = 1$ the number $e^{24\phi_\sigma(z)}$ is an algebraic unit.

In this mémoire, I will give a quick presentation of higher dimensional Arakelov geometry developed in [13, 14, 15, 25, 5, 4, 12, 22] and a generalization of the Siegel function given by Vincent Maillot and Damian Rössler. We begin with an introduction to Gillet and Soulé's *arithmetic intersection theory* and their theory of *characteristic classes for hermitian vector bundles* with values in arithmetic Chow groups. Next, we recall the *arithmetic Riemann Roch theorem*, and its variant for the Adams operations developed by Gillet, Bismut, Soulé and Rössler. They will form the first chapter of this mémoire.

Chapter two is an application of Arakelov theory to abelian schemes, following closely V. Maillot and D. Rössler's paper. We show that there exists a unique class (up to ∂ and $\overline{\partial}$) of currents on the complex points of dual abelian schemes, characterized by three axioms. This class of currents plays the role of the higher Siegel function. In addition, we prove an analogue in Arakelov geometry of a Chern class formula of Bloch and Beauville, in which this class of currents appears. Next, we use the arithmetic Riemann-Roch theorem to show that when restricted to the complement of the zero section, this class of currents can be given by the $(g-1)$ part of the Bismut-Köhler analytic torsion form of the Poincaré bundle along the complement of the zero section. This gives a generalization in higher dimensions of the Kronecker second limit formula. Finally, we apply the Adams-Riemann-Roch theorem to show that when pulling back by a torsion section, the Bismut-Köhler analytic torsion form has a realization in Quillen's algebraic K_1 group of the base. This is a generalization of the property of giving rise to elliptic units. We end the mémoire with an interpretation in the case of dimension one. For further properties of this class of currents, we refer to the original article.

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Chapter 1

A quick tour of Arakelov theory in higher dimensions

In [13], Gillet and Soulé developed an arithmetic intersection theory for arithmetic varieties, generalizing in higher dimensions the works of Arakelov, Faltings, Szpiro and Deligne [1, 9, 10, 8] for arithmetic surfaces. They consider *arithmetic varieties* X, which are regular schemes, quasi-projective and flat over *arithmetic rings* R (for example $R = \mathbb{Z}$, or $R = O_K$), together with complex manifolds $X(\mathbb{C}) = \coprod X_\sigma(\mathbb{C})$ for $\sigma \in \sum$, a finite, linear conjugation-invariant set of embeddings of R into $\mathbb{C}$. An *arithmetic cycle* is defined to be a pair (Z, g) where Z is an algebraic cycle, and g is a real, conjugation-invariant current on $X(\mathbb{C})$ satisfying the equation

$$dd^c g + \delta_{Z(\mathbb{C})} = [w],$$

for w a smooth form. The equivalent classes of arithmetic cycles form the *arithmetic Chow groups*, which have a product structure and functoriality properties. This parallels the development in the algebraic side of *classical Chow groups of varieties over fields* although the arithmetic theory is more complicated due to the analysis involved and the intersection theory is done on schemes over R.

In their following works [14, 15], Gillet and Soulé developped an *arithmetic* theory of characteristic classes for *Hermitian vector bundles*, which are pairs $(\overline{E}, h)$ of an algebraic vector bundle E and a smooth conjugation-invariant hermitian metric h on the holomorphic bundle $E_{\mathbb{C}}$, with values in the arithmetic Chow rings. To each arithmetic cycle (Z, g), there are forgetful maps to Z in classical Chow groups, and to $w = dd^c g + \delta_{Z(\mathbb{C})}$, a form in cohomology class, which is the Poincaré dual of Z. Under those forgetful maps, their arithmetic characteristic classes for hermitian vector bundles become those with values in classical Chow groups, and of Chern-Weil theory with values are closed forms. In general, their arithmetic characters are not additive for exact sequences of hermitian vector bundles. The difference is given by *secondary Bott-Chern classes* [3, 7] which refine Chern-Weil theory at the level of forms.

We recall in the classical situation, if $g : X \to Y$ is a morphism of non-singular varieties, then the Chern character commutes with the pull-back g^* of vector bundles. If g is proper, we can define the direct image map g_* of a vector bundle, framing in the language of Grothendieck groups. However, the Chern character does not commute with g_* and it is described by the Grothendieck-Riemann-Roch theorem. The theorem says that the diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{Td(Tg).ch} & CH^\bullet(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ K_0(Y) & \xrightarrow{ch} & CH^\bullet(Y)_{\mathbb{Q}} \end{array}$$

is commutative: $ch(g_*F) = f_*(ch(F)Td(Tg))$, where $Tg = T_X - g^*T_Y$ is the virtual relative tangent bundle, and F is a vector bundle. In [16], Gillet and Soulé proved an analogue of the Grothendieck-Riemann-Roch theorem for arithmetic varieties. Let F be a vector bundle on X, and $\lambda(F) = \det Rg_*(F)$ be the (algebraic) determinant of cohomology line bundle. Its holomorphic bundle $\lambda(F)_{\mathbb{C}}$ can be endowed with a smooth hermitian metric h, called Quillen metric [3]. The arithmetic Riemann-Roch theorem of Gillet, Soulé and Bismut gives a comparison between $\widehat{c}_1(\det Rg_*(F,h))$ and component (1) of $g_*(\widehat{ch}(F,h))$. In [12], using the higher analytic torsion forms of Bismut-Köhler [3] to define the push-forward of arithmetic K-groups, Gillet, Rössler, and Soulé proved a more general arithmetic Riemann-Roch theorem, stating that the diagram

$$\begin{array}{ccc} \widehat{K_0}(X) & \xrightarrow{\widehat{Td}(Tg).(1-R(Tg_{\mathbb{C}})).\widehat{ch}} & \widehat{CH}^\bullet(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \widehat{K_0}(Y) & \xrightarrow{\widehat{ch}} & \widehat{CH}^\bullet(Y)_{\mathbb{Q}} \end{array}$$

is commutative, where objects with $\widehat{\ }$ are arithmetic generalization, including also differential information as we have seen for the arithmetic Chow groups. The arithmetic push-forward maps $g_* : \widehat{CH}^\bullet(Y)_{\mathbb{Q}} \to \widehat{CH}^\bullet(B)_{\mathbb{Q}}$ and $g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$ involve analysis on the analytic side, integral of currents along the fibres in the first case, and Bismut-Köhler analytic torsion forms in the second case. The exotic cohomology class $R(Tg_{\mathbb{C}})$ is added to make the diagram commutes. This diagram fits in a three-dimensional commutative diagram:

$$
\begin{tikzcd}
K_0(Y) \arrow[rr, "Td(Tg).ch"] \arrow[dd, "g_*"] & & CH(Y)_{\mathbb{Q}} \arrow[dd, dashed, "g_*"] & \\
& \widehat{K}_0(Y) \arrow[ul, two heads] \arrow[rr, "\widehat{Td}(Tg)(1-R(Tg_{\mathbb{C}})).\widehat{ch}"] \arrow[dd, "g_*"] & & \widehat{CH}(Y)_{\mathbb{Q}} \arrow[ul, two heads] \arrow[dd, "g_*"] \\
K_0(B) \arrow[rr, dashed, "ch"] & & CH(B)_{\mathbb{Q}} & \\
& \widehat{K}_0(B) \arrow[ul, two heads] \arrow[rr, "\widehat{ch}"] & & \widehat{CH}(B)_{\mathbb{Q}} \arrow[ul, two heads]
\end{tikzcd}
$$

where various forgetful arrows $\twoheadrightarrow$ are surjective and the diagram behind is the Grothendieck-Riemann-Roch theorem. We will recall the general arithmetic Riemann-Roch theorem, and finish this chapter by mentioning an arithmetic version of Riemann-Roch theorem for Adams operations [22]. Similar to the arithmetic Riemann-Roch theorem, it measures the commutativity of the Adams operations and the push-forward map of arithmetic K-groups.

1.1 Arithmetic intersection theory

The main references for this section are [6, 13, 25, 24].

1.1.1 Currents on complex manifolds

Let X be a smooth quasi-projective complex manifold. Let $D^{p,q}(X)$ and $A^{p,q}(X)$ be spaces of currents and smooth differential forms of type (p,q). By definition, when X is of dimension d, a current of type (p,q) is a linear map

$$S : A_c^{d-p,d-q}(X) \to \mathbb{C}$$

which is continuous for the Schwartz topology.

Example 1.1. Let $Y \subseteq X$ be an analytic cycle of co-dimension p. We can define a current of integration associated to Y in $D^{p,p}(X)$, denoted by δ_Y, by integration on the smooth part of Y :

$$\delta_Y(w) = \int_{Y \setminus Y^{sing}} w.$$

By Hinoranka's works on the resolution of singularities, we can find a proper morphism $\pi : \widetilde{Y} \to Y$ which is an isomorphism on an open dense subset, and $\widetilde{Y}$ is a projective manifold. Then δ_Y is equal to an integral on the compact manifold $\widetilde{Y}$:

$$\delta_Y(w) = \int_{\widetilde{Y}} \pi^*(w)$$

hence converges.

Example 1.2. There is an inclusion $A^{p,q}(X) \subseteq D^{p,q}(X)$, where we consider a form $w \in A^{p,q}(X)$ as a current in $D^{p,q}(X)$ by sending $\eta \in A_c^{d-p,d-q}(X)$ to $\int_X w \wedge \eta$. We denote this current by $[w]$.

Example 1.3. More generally, let $U \subseteq X$ be a dense open set, and w a smooth form on U such that w extends to an L^1- form on X. Then it defines a current $[w] \in D^{p,p}(X)$ by the formula

$$[w](\eta) = \int_U w \wedge \eta$$

1.1.2 Green currents associated to analytic cycles

Similar to forms, we can define the ∂ and $\overline{\partial}$ operators on currents. Let $S \in D^{p,q}(X)$, then its derivatives $\partial S \in D^{p+1,q}(X)$ and $\overline{\partial} S \in D^{p,q+1}(X)$ are defined by

$$\partial S(\eta) = (-1)^{p+q+1} S(\partial \eta)$$

and

$$\overline{\partial} S(\eta) = (-1)^{p+q+1} S(\overline{\partial} \eta)$$

Denote $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/(4\pi i)$, so $dd^c = \frac{\overline{\partial}\partial}{2\pi i}$.

Definition 1. Let Z be a cycle of codimension p on X. A Green current for Z is a current $g \in D^{p-1,p-1}(X)$ such that:

$$dd^c g + \delta_Z = [w]$$

for some smooth form $w \in A^{p,p}(X) \subseteq D^{p,p}(X)$.

Theorem 1.4. *Let X be a compact Kähler manifold, and Z a cycle of codimension p.*
1) There always exists a Green current g_Z for Z. Moreover, g_Z can be taken to be real.
2) If g_1, g_2 are two Green currents for Z, then

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

with $\eta \in A^{p-1,p-1}(X), S_1 \in D^{p-2,p-1}(X), S_2 \in D^{p-1,p-2}(X)$.
3) The current g_Z can be chosen to be of logarithmic type along Z. It means that g_Z is of the form $[g_Y]$ for g_Y a smooth form on $Y = X \setminus Z$, such that g_Y can be extended to an integrable form on X and grows slowly (logarithmic) along Z. In this case, g_Y is called a Green form for the cycle Z.

Example 1.5. In the case $p = 1$, there exists a holomorphic line bundle L, and a section s of L on some dense open set of X such that $Z = div(s)$. Choose a C^∞ Hermitian metric on L. Then the Poincaré - Lelong formula gives

$$dd^c[-\log ||s||^2] + \delta_Z = c_1(L, ||.||)$$

where $[-\log ||s||^2] \in D^{0,0}(X)$ is the current associated to the real-valued L^1 function $-\log ||s||^2$ and $c_1(L, ||.||)$ is the first Chern class of L. By the definition, $[-\log ||s||^2]$ is a Green current for $Z = div(s)$. One can show that all Green currents associated to codimension-1 cycles are obtained in this way. They are also examples of *Green forms of logarithmic type.*

Remark 1.6. The notion of Green forms of logarithmic type is important when we define the pull-back and product of currents. In particular, given two cycles Y and Z meeting properly, and g_Y and g_Z their corresponding currents, we would like to have a Green current for the intersection of Y and Z. Heuristically, we can define $g_Y * g_Z = g_Y \wedge \delta_Z + w_Y \wedge g_Z$ and show that $dd^c(g_Y * g_Z) = -\delta_{Y \cap Z} + w_Y \wedge w_Z$. However, we need to justify the definition of $g_Y \wedge \delta_Z$ and we can do that when the current g_Y is given by a Green form of logarithmic type. The pull-back of currents compatible with cycles is defined through the pull-back of Green forms.

1.1.3 Arithmetic rings

We will study varieties over *arithmetic rings*, which are generalizations of the ring of integers.

Definition 2. An arithmetic ring is a triple $(A, \sum, F_\infty)$ consisting of an excellent regular Noetherian integral domain A, and a finite non-empty set $\sum$ of embeddings $\sigma : A \hookrightarrow \mathbb{C}$, and a conjugate-linear involution of $\mathbb{C}$-algebras, $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \mathbb{C}^{\Sigma} \\ \downarrow{\scriptstyle ||} & & \downarrow{\scriptstyle F_\infty} \\ A & \xrightarrow{\delta} & \mathbb{C}^{\Sigma} \end{array}$$

is commutative. Here $\delta = (\sigma : A \to \mathbb{C})_{\sigma \in \sum}$.

Example 1.7. Let $A = O_K$ be the ring of integers, and $\sum$ be the set of all embeddings of K into $\mathbb{C}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{Z}} A & \xrightarrow{\delta'} & \mathbb{C}^{\Sigma} \\ \downarrow{\scriptstyle c \otimes Id} & & \downarrow{\scriptstyle F_\infty} \\ \mathbb{C} \otimes_{\mathbb{Z}} A & \xrightarrow{\delta'} & \mathbb{C}^{\Sigma} \end{array}$$

where $c(z) = \overline{z}$ and $\delta' = \{Id \otimes \sigma\}_{\sigma \in \sum}$. We remark that $[O_K : \mathbb{Z}] = [K : Q] =$ # of field embeddings of $K \to \mathbb{C}$, and δ' is an isomorphism. We can take F_∞ to be the involution induced by $c \otimes Id$.

Example 1.8. Let $A = \mathbb{C}$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{R}} A & \xrightarrow{\delta'} & \mathbb{C} \times \mathbb{C} \\ \downarrow{\scriptstyle c \otimes Id} & & \downarrow{\scriptstyle F_\infty} \\ \mathbb{C} \otimes_{\mathbb{Z}} A & \xrightarrow{\delta'} & \mathbb{C} \times \mathbb{C} \end{array}$$

where $\delta' : z \otimes w \to (zw, z\overline{w})$ is an isomorphism. The embedding of $A \to \mathbb{C} \times \mathbb{C}$ is given by the composition $A \to \mathbb{C} \otimes_{\mathbb{R}} A \to \mathbb{C} \times \mathbb{C} : a \to 1 \otimes a \to (a, \overline{a})$. We can take $\sum$ to be $\{Id, c\}$ where c is the complex conjugation, and F_∞ is the involution induced from $c \otimes Id$. The map F_∞ sends $(zw, z\overline{w}) \longrightarrow (\overline{z}w, \overline{z\overline{w}})$, and in particular, $F_\infty(a, b) = (\overline{b}, \overline{a})$.

1.1.4 Arithmetic Chow groups and arithmetic cycles

Let $(A, \sum, F_\infty)$ be an arithmetic ring, and F its fraction field.

Definition 3. An arithmetic variety X is a regular scheme, quasi-projective and flat over A, with smooth generic fibre X_F.

Let X be an arithmetic variety. Because X is quasi-projective and X_F is smooth, $X(\mathbb{C}) = \oplus_{\sigma \in \sum} X \otimes_\sigma \mathbb{C} = X \otimes_A \mathbb{C}^\Sigma$ is a quasi-projective complex manifold. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation induced from $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$. We denote $D^{p,p}(X_\mathbb{R}) \subseteq D^{p,p}(X(\mathbb{C}))$ (resp. $A^{p,p}(X_\mathbb{R}) \subseteq A^{p,p}(X(\mathbb{C}))$) the sub-spaces of real currents (resp. forms) T such that $F_\infty^* T = (-1)^p T$. We define

$$\widetilde{D}^{p,p}(X_\mathbb{R}) = D^{p,p}(X_\mathbb{R})/(im\partial + im\overline{\partial})$$

(resp.

$$\widetilde{A}^{p,p}(X_\mathbb{R}) = A^{p,p}(X_\mathbb{R})/(im\partial + im\overline{\partial}) \qquad).$$

Definition 4. Let Z be an algebraic cycle of codimension p on X. A Green current for Z is a Green current g_Z for the associated cycle $Z(\mathbb{C})$ on $X(\mathbb{C})$ such that g_Z lies in $D^{p-1,p-1}(X_\mathbb{R})$.

Remark 1.9. From our discussion of currents on complex manifolds, there always exists a current associated to a cycle Z.

Definition 5. An arithmetic cycle is a pair (Z, g) where Z is an algebraic cycle on X, and g is a Green current for Z.

Example 1.10. The pairs $(0, \partial u + \overline{\partial} v)$ where $u \in D^{p-2,p-1}(X(\mathbb{C}))$ and $v \in D^{p-1,p-2}(X(\mathbb{C}))$, and $\partial u + \overline{\partial} v \in D^{p-1,p-1}(X_\mathbb{R})$ are arithmetic cycles.

Example 1.11. Let $X^{(p-1)}$ be the set of points of co-dimension $p-1$ on X. If $x \in X^{(p-1)}$ then $Y = \overline{\{x\}}$ is a closed irreducible sub-scheme of X of co-dimension $p-1$. For $f \in k(x)^*$, a rational function on Y different from 0, there is an arithmetic cycle $(div(f), [-log|f|^2] \ \ \delta_{Y(\mathbb{C})})$ where $div(f)$ is the divisor of f (a cycle of co-dimension p on X) and $[-log|f|^2] \ \ \delta_{Y(\mathbb{C})} \ \in D^{p-1,p-1}(X_\mathbb{R})$ is a current that maps w to $-\int_{Y(\mathbb{C})}(log|f|^2)w$. The integral converges by resolution of singularities.

Definition 6. The *p-th* arithmetic Chow group $\widehat{CH}^p(X)$ is the group generated by arithmetic cycles of co-dimension p, modulo the subgroup defined by the cycles we just defined.

$$\widehat{CH}^p(X) = \frac{\{(Z,g), codim(Z)=p\}}{\{(0, \partial u + \overline{\partial} v), (div(f), [-log|f|^2] \quad \delta_{Y(\mathbb{C})}, f \in X^{(p-1)})\}}$$

Example 1.12. Let $\overline{L} = (L,h)$ be an Hermitian line bundle on X, i.e. L is an algebraic line bundle on X, and h is a C^∞ Hermitian metric on the corresponding holomorphic line bundle $L_{\mathbb{C}}$ on $X(\mathbb{C})$ that is invariant by complex conjugation: $F_\infty^*(h) = h$. Choose a rational section s of X (defined on a dense open set of X). Then the pair (div $s, [-log||s||^2]$) is an arithmetic cycle of co-dimension one. Its class $\hat{c}_1(L, ||.||) \in \widehat{CH}^1(X)$ is independent of the choice of the section s. It is called the first Chern class of $\overline{L} = (L,h)$. One can show that any element of $\widehat{CH}^1(X)$ is of this form. If we define $\widehat{Pic}(X)$ to be the group of isomorphism classes of hermitian line bundles, with the group structure

$$(L,h).(L,h') = (L \otimes L', h \otimes h')$$

and $||s \otimes s'|| = ||s||.||s'||$ then

Proposition 1.13. *The first Chern class $\hat{c}_1$ induces a group isomorphism*

$$\hat{c}_1 : \widehat{Pic}(X) \cong \widehat{CH}^1(X)$$

1.1.5 Fundamental exact sequences

We define

$$\widehat{CH}^\bullet(X) = \oplus_p \widehat{CH}^p(X)$$

$$Z^{p,p}(X_{\mathbb{R}}) = \ker(d : A^{p,p}(X_{\mathbb{R}}) \to A^{2p+1}(X(\mathbb{C}))) \subseteq A^{p,p}(X_{\mathbb{R}})$$

$$H^{p,p}(X_{\mathbb{R}}) = \{c \in H^{p,p}(X(\mathbb{C})) : c \text{ real }, F_\infty^* c = (-1)^p c\} \subseteq \widetilde{A}^{p,p}(X_{\mathbb{R}})$$

Proposition 1.14. *We have the following exact sequences of abelian groups:*

$$CH^{p,p-1}(X) \xrightarrow{\rho} H^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta,\omega} CH^p(X) \oplus Z^{p,p}(X_{\mathbb{R}}) \xrightarrow{cl} H^{p,p}(X_{\mathbb{R}}) \to 0$$

$$CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

$CH^{p,p-1}(X)$ is a motivic cohomology group, appearing in Quillen's spectral sequence. If X is Noetherian,

$$CH^{p,p-1}(X) \cong \frac{\{f_y \in \oplus_{y \in X^{(p-1)}} k(y)^* : \sum_y div(f_y) = 0\}}{\{d_1(\{u_z\}) : (\{u_z\}) \in \oplus_{z \in X^{(p-2)}} K_2(k(z))\}}$$

The term $K_2(k(z))$ is Quillen's algebraic K_2-group. The map $d_1 : \oplus_{z \in X^{(p-2)}} K_2(k(z)) \to \oplus_{y \in X^{(p-1)}} k(y)^*$ is given by the tame symbol.

The morphisms in the theorem are defined as follows:

- An element of $CH^{p,p-1}(X)$ is represented by $(f_y)_y, (f_y \in k(y)^*, y \in X^{(p-1)})$ such that $\sum_y div(f_y) = 0$. The current $\sum_y -[\log |f_y|^2] \in D^{p-1,p-1}(X_{\mathbb{R}})$ satisfies

  $$dd^c(\sum_y -[\log |f_y|^2]) = -\delta_{\sum_y div(y)} = 0$$

  hence defines an element in $H^{p-1,p-1}(X_{\mathbb{R}})$, and also in $\widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$, which we denote $\rho((f_y)_y)$.

- Let $cl(\eta)$ denote the class of $\eta \in \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$. We define

  $$a : \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \to \widehat{CH}^p(X)$$

  $$cl(\eta) \to [(0, [\eta])]$$

- $$\zeta : \widehat{CH}^p(X) \to CH^p(X)$$

  $$[(Z, g_Z)] \to [Z]$$

- $$\omega : \widehat{CH}^p(X) \to ker d \cap ker d^c (\subseteq Z^{p,p}(X_{\mathbb{R}}))$$

  $$[(Z, g_Z)] \to w_Z$$

  where $dd^c g_Z + \delta_Z = [w_Z]$.

- The map cl: $CH^p(X) \oplus Z^{p,p}(X_{\mathbb{R}}) \to H^{p,p}(X_{\mathbb{R}})$ is given by

  $$cl([Z], w) = cl(Z) - cl(w)$$

  where cl(Z) is the class of Z in $H^{p,p}(X_{\mathbb{R}})$ and $cl(w)$ is the image of w via the projection $Z^{p,p}(X_{\mathbb{R}}) \to H^{p,p}(X_{\mathbb{R}})$ sending a closed form to its cohomology class.

Remark 1.15. By definition,

$$a(\eta)x = a(\eta.w(x))$$

for any $x \in \widehat{CH}^q(X)$ and $\eta \in \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$.

Remark 1.16. In the exact sequence

$$CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

the map ρ is the composition of the following:

$$CH^{p,p-1}(X) \xrightarrow{cyc} H_D^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{forgetful} H_{D,an}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \hookrightarrow \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$$

where *cyc* is the cycle class map and $forgetful$ is the forgetful map from real Deligne Beilinson cohomology to real analytic Deligne cohomology. For the definition of real Deligne Beilinson cohomology and analytic Deligne cohomology, see [19, 11].

Example 1.17. Consider the exact sequence

$$CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

When $p = 1$, $CH^{0,1}(X) = \mathscr{O}(X)^*$ and $\widetilde{A}^{0,0}(X_{\mathbb{R}}) = A^{0,0}(X_{\mathbb{R}}) = C^\infty(X(\mathbb{C}), \mathbb{R})^{F_\infty}$, the space of real C^∞ functions invariant under complex conjugation. The sequence becomes

$$\mathscr{O}(X)^* \xrightarrow{(-\log|\ \ |_\sigma^2)_{\sigma\in\sum}} C^\infty(X(\mathbb{C}), \mathbb{R})^{F_\infty} \xrightarrow{a} \widehat{CH}^1(X) \xrightarrow{\zeta} CH^1(X) \to 0$$

Moreover, if $X = Spec(A)$ and $A = O_F$, where F is a number field, the sequence becomes

$$1 \to \mu(F) \to A^* \xrightarrow{(-\log|\ \ |_\sigma^2)_{\sigma\in\sum}} (\oplus_{\sigma\in\sum}\mathbb{R}) \xrightarrow{a} \widehat{CH}^1(X) \xrightarrow{\zeta} Cl(O_F) \to 0$$

where $\mu(F)$ is the group of roots of unity in F.

1.1.6 Functoriality and multiplicative structures

Let $A = (A, \sum, F_\infty)$ be an arithmetic ring, and F the fraction field of A.

Theorem 1.18. *1) Let $f : X \to Y$ be a morphism of arithmetic varieties over A , and suppose that f induces a smooth map $X_F \to Y_F$ between generic fibres of X and Y, and that f is flat. Then it determines a pull-back morphism of abelian groups*

$$f^* : \widehat{CH}^*(Y) \to \widehat{CH}^*(X)$$

2) Given two flat maps $f : X \to Y$ and $g : Y \to Z$ inducing smooth maps between generic fibers, then $f^* g^* = (gf)^*$.

3) The pull-back morphism is compatible with $CH^{p,p-1}, CH^p, \ldots$, and with the fundamental exact sequences.

We give a construction of the map f^*: If $Z = \sum n_i [Z_i] \in Z^p(Y)$, then $f^*[Z] = \sum n_i [f^{-1} Z_i]$ is a co-dimension p cycle on X. Since $f : X_F \to Y_F$ is smooth, $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a submersion. For any current $T \in D^{p,q}(Y_{\mathbb{R}})$, we can define $f_{\mathbb{C}}^* T(\phi) = T(f_{\mathbb{C},*}\phi)$, where ϕ is a compactly supported form on $X_{\mathbb{C}}$. The map $f_{\mathbb{C},*} : A_c^{p,q}(X_{\mathbb{C}}) \to A_c^{p-d,q-d}(Y_{\mathbb{C}})$ is integration over the fibre.

Proposition 1.19. *Suppose X and Y are equidimensional.*

1) If $f : X \to Y$ is a proper morphism, of relative dimension $d = \dim(X) - \dim(Y)$, such that f induces a submersion $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$, then it determines a push-forward morphism of abelian groups $f_* : \widehat{CH}^*(X) \to \widehat{CH}^{*-d}(Y)$.

2) Given two maps $f : X \to Y$ and $g : Y \to Z$, then $g_* f_* = (gf)_*$.

3) The push-forward morphism is compatible with a, ζ, ω.

4) The projection formula holds:

$$f_*(x.f^*(y)) = f_*(x).y$$

We construct the map f_* by: $f_*((Z, g_Z)) = (f_*(Z), f_*(g_Z))$, where $f_*(g_Z)(w) = g_Z(f^*(w))$. To define $f_*(Z)$, by linearity, we can assume that Z is an irreducible subset of X. Let $f(Z)$ be its image. If $f(Z)$ has the same dimension as Z, define $f_*(Z) = n.Z$, where n is the degree of the extension of fields $k(Z)$ over $k(f(Z))$. If $f(Z)$ has a dimension less than $\dim(Z)$, define $f_*(Z) = 0$.

Proposition 1.20. *There exists an associative commutative graded bilinear pairing*

$$\widehat{CH}^p(X) \times \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that is compatible with ζ and the multiplicative structure on $CH^\bullet(X)_{\mathbb{Q}}$ defined by intersection of classes of algebraic cycles, and compatible with ω and multiplication of differential forms. It is also compatible with inverse image, i.e. $f^*(xy) = f^*(x) f^*(y)$.

The difficulty when defining the product of arithmetic cycles is that there is no-known version of Chow's moving lemma for varieties over $\mathbb{Z}$. To define the product of two algebraic cycles, Gillet and Soulé used an isomorphism

$$CH^p(X)_{\mathbb{Q}} \cong K_0(X)^{(p)}$$

where $K_0(X)^{(p)} \subseteq K_0(X)$ is a subspace where each Adams operation $\psi^k, k \geq 1$ acts by multiplication by k^p. The pairing between $K_0(X)^{(p)}$ and $K_0(X)^{(q)}$ is given by tensor product of O_X- modules. Another difficulty is to define a product of currents associated to an intersection of two algebraic cycles. This can be done using Green forms of logarithmic type as mentioned before.

1.2 Arithmetic characteristic classes for Hermitian vector bundles

The main references for this sections are [14, 15].

1.2.1 Secondary Bott Chern classes

Let X be a complex manifold and $\overline{E} = (E, h)$ a holomorphic vector bundle of rank n on X, endowed with a hermitian metric h. For any symmetric formal series $\phi \in \mathbb{Q}[[T_1, T_2, ..., T_n]]$ and $k \geq 0$, let $\phi^{(k)}$ be the homogeneous component of ϕ of degree k. We identify $\phi^{(k)}$with $\phi^{(k)} : M_n(\mathbb{C}) \to \mathbb{C}$, the unique polynomial map which is invariant under conjugation by $GL_n(\mathbb{C})$, and such that its value on a diagonal matrix $\mathrm{diag}(\lambda_1, ..., \lambda_n)$, $\lambda_i \in \mathbb{C}$ is equal to $\phi^{(k)}(\lambda_1, ..., \lambda_n)$. By Chern-Weil theory, we can associate to ϕ and E a sum of d and d^c closed forms $\phi(\overline{E}) = \phi(-K/(2\pi i))$ in $\oplus_{p\geq 0} A^{p,p}(X)$, where K is the curvature of a (Chern) connection on E. We recall that a curvature is a 2-form with value in $End(E)$ and we can identify $End(E)$ with $M_n(\mathbb{C})$ locally, and use ϕ to evaluate K. The evaluation makes sense because ϕ is invariant by conjugation. The form $\phi(\overline{E})$ satisfies:

- The de Rham cohomology class of $\phi(\overline{E})$ is independent of the metric h.
- $f^*\phi(\overline{E}) = \phi(f^*\overline{E})$ for every holomorphic map $f : Y \to X$.
- $\phi(\overline{E} \oplus \overline{F}) = \phi(\overline{E}) + \phi(\overline{F})$.

The Chern and Todd classes are defined using the series

$$ch(T_1, T_2, .., T_n) = \sum_{i=1}^{n} exp(T_i)$$

$$Td(T_1, T_2, ..., T_n) = \prod_{i=1}^{n} (T_i/(1 - exp(-T_i)))$$

The cohomology class of $\phi(\overline{E})$ is independent of the choice of a hermitian metric on E, but not at the level of forms. This idea is made precisely by the existence of secondary characteristic classes.

Theorem 1.21. *Let $\phi \in \mathbb{Q}[[T_1, T_2, ..., T_n]]$ be a symmetric formal series, and $\overline{\mathcal{E}}$ a short exact sequence of Hermitian vector bundles over a complex manifold X:*

$$\overline{\mathcal{E}} : 0 \to \overline{S} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{Q} \to 0$$

with rank $E = n$. Then there is a unique way to attach to $\overline{\mathcal{E}}$ a class of forms $\widetilde{\phi}(\overline{\mathcal{E}}) \in \widetilde{A}(X) = \oplus_{p\geq 0} A^{p,p}(X)/(im\partial + im\overline{\partial})$ such that:
i) $dd^c\widetilde{\phi}(\overline{\mathcal{E}}) = \phi(\overline{S} \oplus \overline{Q}) - \phi(\overline{E})$.
ii) $\widetilde{\phi}(\overline{\mathcal{E}})$ commutes with pull-back: for all morphisms $f : Y \to X$ of complex manifolds,

$$\widetilde{\phi}(f^*(\overline{\mathcal{E}})) = f^*(\widetilde{\phi}(\overline{\mathcal{E}}))$$

iii) If $\overline{\mathcal{E}}$ is of the form

$$0 \to \overline{S} \xrightarrow{i} \overline{S} \oplus \overline{Q} \xrightarrow{p} \overline{Q} \to 0$$

where $i(x) = x \oplus 0$ and $p(x \oplus y) = y$ then $\widetilde{\phi}(\overline{\mathcal{E}}) = 0$.

Remark 1.22. Because $dd^c(\partial + \overline{\partial}) = 0$, it makes sense to define secondary characteristic classes in $\oplus_{p\geq 0} A^{p,p}(X)/(im\partial + im\overline{\partial})$. We also remark that the secondary characteristic classes can be defined for any long exact sequence of finite terms.

Definition 7. The secondary classes associated to the Chern character's series is called the secondary Bott-Chern classes.

Proposition 1.23. *The Bott-Chern secondary classes satisfy*

- $\widetilde{ch}(\overline{\mathcal{E}}_1 \oplus \overline{\mathcal{E}}_2) \equiv \widetilde{ch}(\overline{\mathcal{E}}_1) + \widetilde{ch}(\overline{\mathcal{E}}_2)$
- $\widetilde{ch}(\overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2) \equiv \widetilde{ch}(\overline{\mathcal{E}}_1).\widetilde{ch}(\overline{\mathcal{E}}_2)$

- *If we have a symmetry:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'_3 & \longrightarrow & E_3 & \longrightarrow & E''_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all rows $\mathcal{E}_i$ and all columns $\mathcal{F}_j$ are exact ($i, j = 1, 2, 3$), then

$$\sum_{i=1}^{3}(-1)^i\widetilde{ch}(\overline{\mathcal{E}}_i) \equiv \sum_{j=1}^{3}(-1)^j\widetilde{ch}(\overline{\mathcal{F}}_j)$$

1.2.2 Arithmetic characteristic classes

Let X be an arithmetic variety over an arithmetic ring $(A, \sum, F_\infty)$.

Definition 8. An hermitian vector bundle $\overline{E} = (E, h)$ on X is an algebraic vector bundle E on X such that the induced holomorphic vector bundle $E_{\mathbb{C}}$ on $X(\mathbb{C})$ has an hermitian metric h invariant under complex conjugation, i.e. $F_\infty^*(h) = h$.

Theorem 1.24. *For all Hermitian vector bundle $\overline{E}$ of rank n over an arithmetic variety X, and for all symmetric series $\phi \in \mathbb{Q}[[T_1, T_2, ..., T_n]]$, we can associate a characteristic class $\hat{\phi}(\overline{E}) \in \widehat{CH}^{\bullet}(X)_{\mathbb{Q}}$ satisfies the following conditions:*
1) **Functoriality**. *If $f : Y \to X$ is a morphism of arithmetic varieties, and $\overline{E}$ is a Hermitian vector bundle over X then*

$$f^*(\hat{\phi}(\overline{E})) = \hat{\phi}(f^*(\overline{E}))$$

2) **Normalization**. *If $\overline{E} = \overline{L_1} \oplus \overline{L_2} \oplus .... \oplus \overline{L_n}$ is the orthogonal direct sum of Hermitian line bundles then*

$$\hat{\phi}(\overline{E}) = \hat{\phi}(\hat{c}_1(\overline{L_1}), \hat{c}_1(\overline{L_2}), ..., \hat{c}_1(\overline{L_n}))$$

3) **Twist by a line bundle**. *Let* $\phi_i \in \mathbb{Q}[[T_1, T_2, ..., T_n]]$ *satisfy*

$$\phi(T_1 + T, ...T_n + T) = \sum_{i \geq 0} \phi_i(T_1, ..., T_n)T^i$$

If $\overline{E}$ *(resp.* $\overline{L}$*) is a Hermitian vector bundle (resp. line bundle), then*

$$\hat{\phi}(\overline{E} \oplus \overline{L}) = \sum_i \hat{\phi}_i(\overline{E})\hat{c}_1(\overline{L})^i$$

4) **Compatibility with characteristic forms**. *For all Hermitian vector bundles* $\overline{E}$ *over* X*,*

$$\omega(\hat{\phi}(\overline{E})) = \phi(\overline{E_{\mathbb{C}}}) \in \oplus_{p \geq 0} A^{p,p}(X)$$

$$\zeta(\hat{\phi}(\overline{E})) = \phi(E) \in CH^{\bullet}(X)_{\mathbb{Q}}$$

5) **Compatible with short exact sequences**. *For all short exact sequence*

$$\overline{\mathcal{E}} : 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$$

of Hermitian vector bundles on X*,*

$$\hat{\phi}(\overline{S} \oplus \overline{Q}) - \hat{\phi}(\overline{E}) = a(\widetilde{\phi}(\overline{\mathcal{E}}))$$

Moreover, properties 1),2),3),4) characterize this construction.

Theorem 1.25. *Let* $\overline{E}$ *and* $\overline{F}$ *be Hermitian vector bundles over an arithmetic variety* X*. Then*

$$\hat{c}(\overline{E} \oplus \overline{F}) = \hat{c}(\overline{E})\hat{c}(\overline{F}), \qquad \widehat{ch}(\overline{E} \oplus \overline{F}) = \widehat{ch}(\overline{E}) + \widehat{ch}(\overline{F}),$$

$$\widehat{ch}(\overline{E} \otimes \overline{F}) = \widehat{ch}(\overline{E})\widehat{ch}(\overline{F}), \qquad \widehat{Td}(\overline{E} \oplus \overline{F}) = \widehat{Td}(\overline{E}).\widehat{Td}(\overline{F}),$$

$$\widehat{ch}(\overline{E})^{(1)} = \hat{c}_1(\overline{E}) \text{ in } \widehat{CH}^1(X)_{\mathbb{Q}}$$

$$\text{For } p \geq rank(E), \hat{c}_p(\overline{E}) = 0$$

1.3 Arthmetic Riemann-Roch theorem in higher degrees

The main references for this section are [21, 5, 12].

1.3.1 Arithmetic K-groups

Let X be an arithmetic variety over an arithmetic ring $(A, \sum, F_\infty)$.

Definition 9. The arithmetic Grothendieck group $\widehat{K}_0(X)$ associated to X is the abelian group generated by $\widetilde{A}(X_{\mathbb{R}}) = \oplus_{p\geq 0}\widetilde{A}^{p,p}(X_{\mathbb{R}})$ and the isometry classes of hermitian vector bundles on X, with the relations

$$\widetilde{ch}(\overline{\mathcal{E}}) = \overline{E'} - \overline{E} + \overline{E}'' \text{ for every exact sequence } \overline{\mathscr{E}} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0.$$

$$\eta = \eta' + \eta'', \text{ if } \eta \in \widetilde{A}(X) \text{ is the sum of two elements } \eta' \text{ and } \eta''.$$

Theorem 1.26. *There is an exact sequence of abelian groups*

$$K_1(X) \xrightarrow{\rho} \widetilde{A}(X_{\mathbb{R}}) \xrightarrow{\alpha} \widehat{K}_0(X) \xrightarrow{\beta} K_0(X) \to 0$$

where $\alpha(\eta) = [(0, \eta)]$ and $\beta([\overline{E}, \eta]) = [E]$ where $\eta \in \widetilde{A}(X_{\mathbb{R}})$ and $\overline{E}$ is an hermitian vector bundle. The group $K_1(X)$ is Quillen algebraic K_1 group, and $H^{2p-1}_{D,an}(X_{\mathbb{R}}, \mathbb{R}(p))$ is real Deligne cohomology group.The map $\rho : K_1(X) \to \oplus_{p\geq 1} H^{2p-1}_{D,an}(X_{\mathbb{R}}, \mathbb{R}(p)) \subseteq \widetilde{A}(X_{\mathbb{R}})$ is $(-2)\times$ the Beilinson regulator map.

Lemma 1.27. *There is a unique ring homomorphism $\widehat{ch} : \widehat{K}_0(X) \to \widehat{CH}^{\bullet}(X)_{\mathbb{Q}}$, commuting with pull-back maps, such that*

- *The formula $\widehat{ch}(\eta) = (0, \eta)$ holds, if $\eta \in \widetilde{A}(X_{\mathbb{R}})$.*
- *The formula $\widehat{ch}(\overline{L}) = exp(\widehat{c}_1(\overline{L}))$ holds, if $\overline{L} = (L, h^L)$ is a hermitian line bundle on X.*
- *The formula $\omega(\widehat{ch}(E)) = ch(\overline{E})$ holds for any hermitian vector bundle $\overline{E}$ on X.*

1.3.2 Bismut-Köhler nalytic torsion forms

Ray-Singer holomorphic torsion

Let X be a compact complex manifold of dimension n, and w is a Kähler form on X. Let $\overline{E} = (E, h)$ be a hermitian vector bundle on X. The form w determines a hermitian metric g on the holomorphic tangent vector bundle T_X and a volume form μ on X, characterized by

$$w = \frac{i}{2\pi} \sum_{\alpha,\beta=1}^{n} g(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}) dz_\alpha \wedge d\overline{z}_\beta$$

and

$$\mu = \frac{1}{n!} w^n$$

Consider the Dolbeault complex

$$... \to A^{0,q}(X,E) \xrightarrow{\overline{\partial}_E} A^{0,q+1}(X,E) \to ...$$

where $A^{p,q}(X,E)$ is the vector space of smooth forms of type (p,q) with coefficients in E, and $\overline{\partial}_E$ is the Cauchy-Riemann operator. For each q, we can define an L^2 metric on $A^{p,q}(X,E)$ by the formula

$$< s_1, s_2 >_{L^2} = \int_X < s_1(x), s_2(x) > \mu$$

where $< s_1(x), s_2(x) >$ is the point-wise scalar product coming from the metric on E and the metric on differential forms induced from w. The operator $\overline{\partial}_E$ admits an adjoint $\overline{\partial}_E^*$ for this scalar product. For $s_1 \in A^{0,q}(X,E), s_2 \in A^{0,q+1}(X,E)$

$$< \overline{\partial}_E s_1, s_2 >_{L^2} = < s_1, \overline{\partial}_E^* s_2 >_{L^2}$$

Let $\Delta_{q,E} = \overline{\partial}_E \overline{\partial}_E^* + \overline{\partial}_E^* \overline{\partial}_E$ be the Laplace operator on $A^{0,q}(X,E)$ and $\mathcal{H}^{0,q}(X,E) = Ker \Delta_{q,E}$ be the set of harmonic forms. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ....$ be the eigenvalues of $\Delta_{q,E}$ on the orthogonal complement to $\mathcal{H}^{0,q}(X,E)$ indexed in an increasing order and taken into account multiplicities. They are positive. We can define the Dirichlet series

$$\zeta_q(s) = \sum_{n \geq 1} \lambda_n^{-s}$$

For $\lambda > 0$, $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} dt$, hence

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n \geq 1} e^{-\lambda_n t} dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-t\Delta_{q,E}} - P_q) dt$$

where P_q is the projection of $A^{0,q}(X,E)$ to the subspace $\mathcal{H}^{0,q}(X,E)$. By studying the asymptotic behavior of the function $\theta(t) = \sum_{n \geq 1} e^{-\lambda_n t}$ when $t \to 0$ and $t \to +\infty$ using heat kernel method ([25], page 98), one can show that when $Re(s) > \dim_{\mathbb{C}} X$, $\zeta_q(s)$ converges absolutely, and has a meromorphic continuation to the whole complex plane which is holomorphic at 0. The Ray-Singer holomorphic torsion is defined as

$$T^{RS}(X, w, \overline{E}) = \sum_{q \geq 0}^{\dim_{\mathbb{C}} X} (-1)^{q+1} q \zeta_q'(0)$$

Ray-Singer holomorphic torsion for flat line bundles on the torus

Let $\Gamma = \mathbb{Z} + \mathbb{Z}.\tau$ be a lattice in $\mathbb{C}$, $X = \mathbb{C}/\Gamma$ be a torus, and $\chi : \Gamma \cong \pi_1(W) \to S_1$ a non-trivial unitary one dimensional representation of $\pi_1(X)$. We can identity χ with an element in the Picard variety $Pic^0(X)$ of X. If χ is given by $\chi(m\tau + n) = e^{2\pi i(mu+nv)}$, then the corresponding element in $Pic^0(X)$ is given by $u - \tau v$. Moreover, in this case, $\zeta_0(s) = \zeta_1(s)$, hence

$$T^{RS}(W, u - \tau v) = \zeta'(0)$$

where $\zeta(s) = \zeta_0(s)$ is the zeta function for the Laplacian on the space $A^{0,0}(X, u - \tau v)$ of C^∞ sections of the flat line bundle associated to χ. Explicitly in terms of u, v, τ, the eigenvalues of Δ on $A^{0,0}(X, u - \tau v)$ are:

$$\lambda_{m,n} = -\frac{4\pi^2}{(Im\tau)^2}|u + m - \tau(v + n)|^2$$

By explicit formula for the heat kernel, or by the Poisson summation formula relating the theta functions of Γ and its dual lattice, we have:

$$tr(e^{-t\Delta}) = \theta(t) = \sum_{m,n} e^{-\lambda_{m,n}t} = \sum_{m,n} \frac{Im\ \ \tau}{4\pi t} e^{-|m\tau+n|^2/(4t)} e^{2\pi i(mu+nv)}$$

This gives the asymptotic expansion $tr(e^{-t\Delta}) \sim \frac{Im\tau}{4\pi t}$ when $t \to 0$, hence the analytic continuation of the zeta function. For $Re(s)$ large, we can write

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} tr(e^{-t\Delta}) dt = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} tr(e^{-t\Delta}) dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} tr(e^{-t\Delta}) dt$$

$$= \frac{1}{\Gamma(s)} \frac{Im\tau}{4\pi(s-1)} + \frac{Im\tau}{4\pi\Gamma(s)} \sum_{(m,n)\neq(0,0)} e^{2\pi i(mu+nv)} \int_0^1 t^{s-2} e^{-|m\tau+n|^2/(4t)} dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} tr(e^{-t\Delta}) dt$$

It defines a meromorphic function in the complex plane, which vanishes at $s = 0$. When $Re(s) < 0$, we can plug in the expression for $tr(e^{-t\Delta})$ to the last integral to obtain:

$$\zeta(s) = \frac{Im\tau}{4\pi\Gamma(s)} \sum_{(m,n)\neq(0,0)} e^{2\pi i(mu+nv)} \int_0^\infty t^{s-2} e^{-|m\tau+n|^2/(4t)} dt$$

$$= \frac{y}{4\pi} \frac{\Gamma(1-s)}{\Gamma(s)} \sum_{(m,n)\neq(0,0)} e^{2\pi i(mu+nv)} \left(\frac{4}{|m\tau+n|^2}\right)^{1-s}$$

We assume $v \neq 0 \pmod 1$. The case $u \neq 0 \pmod 1$ is similar. Following Siegel [23], by breaking the sum above over $m = 0, n \neq 0$ and $m \neq 0$, we can show that $\zeta(s)$ given by the last expression converges uniformly for $Re(s) < 1/2$. Thus, we can evaluate $\zeta'(s)$ at 0 to get:

$$\zeta'(0) = \frac{1}{\pi} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(mu+nv)}.y}{|m\tau+n|^2}$$

where $\tau = x+iy$. By applying Kronecker's second limit formula, $\zeta'(0) = \frac{1}{\pi}.(-2\pi)\log|g_{-v,u}(\tau)| = -2\log|g_{-v,u}(\tau)| = -2\log|g_{u,-v}(\tau)|$. Thus

$$T^{RS}(X, u-\tau v) = -2\log|g_{u,-v}(\tau)|$$

Bismut-Köhler higher analytic torsion form

Let $\pi : M \to B$ be a holomorphic submersion of complex manifolds with compact fibre Z. Let TM, TB be the holomorphic tangent bundles of M and B. Let $TZ = TM/B$ be the holomorphic tangent bundle to the fibre Z. Let g^{TZ} be a Hermitian metric on TZ. Let T^HM be a vector subbundle of TM such that

$$TM = T^HM \oplus TZ$$

Definition 10. The triple (π, g^{TZ}, T^HM) defines a Kähler fibration if there exists a smooth real (1,1)-form w over M, with the following properties:

- w is closed.
- $T^H_{\mathbb{R}}M$ and $T_{\mathbb{R}}Z$ are orthogonal with respect to w.
- The form w induces a Kähler metric on $T_{\mathbb{R}}Z$, i.e. if $X, Y \in T_{\mathbb{R}}Z$, then

  $$w(X,Y) = < X, J^{TZ}Y >_{g^{TZ}}$$

  where J^{TZ} is the complex structure on the real tangent bundle $T_{\mathbb{R}}Z$.

Assume that we have a $(1,1)$ form w on M inducing a Kähler fibration. Let ζ be a holomorphic vector bundle on M. Let h^ζ be a Hermitian metric on ζ. We assume that the sheaves $R^k\pi_*\zeta$'s are locally free. For example, if $R^j\pi_*\zeta = 0$ for $j \geq 1$, then $\pi_*\zeta$ is locally free by the semi-continuity theorem. By Hodge theory, there are isomorphisms between the fibres of $R^j\pi_*\zeta$ and the corresponding harmonic forms in the relative Dolbeault complex $\Omega^\bullet(Z, \zeta_{|Z})$. The harmonic forms inherit the L_2 metric, coming from g^{TZ} and h^ζ. Hence the metrics g^{TZ} and h^ζ determine metrics $h^{R^j\pi_*\zeta}$ on the vector bundles $R^j\pi_*\zeta$'s. Consider

$$ch(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}) = \sum_{j=0}(-1)^j ch(R^j\pi_*\zeta, h^{R^j\pi_*\zeta})$$

and the secondary Bott-Chern class $\widetilde{ch}(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}, h'^{R^\bullet\pi_*\zeta})$ which satisfies:

$$dd^c\widetilde{ch}(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}, h'^{R^\bullet\pi_*\zeta}) = ch(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}) - ch(R^\bullet\pi_*\zeta, h'^{R^\bullet\pi_*\zeta})$$

The Bismut-Köhler analytic torsion form $T(w, h^\zeta) \in \oplus_{p\geq 0}\widetilde{A}^{p,p}(B_\mathbb{R})$ satisfies the following conditions:

- It solves the transgression formula:

$$dd^cT(w, h^\zeta) = ch(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}) - \int_Z Td(TZ, g^{TZ})ch(\zeta, h^\zeta).$$

  This equation makes precise the Grothendieck-Riemann Roch theorem for submersion at the level of cohomology class.

- Its component in degree 0 coincides with a function on B, which calculates the Ray-Singer holomorphic torsions along the fibres.

- Modulo ∂ and $\overline{\partial}$, $T(w, h^\zeta)$ depends only on the metrics (g^{TZ}, h^ζ).

- If (w', h'^ζ) is another set of data similar to (w, h^ζ) then

$$T(w', h'^\zeta) - T(w, h^\zeta) = \widetilde{ch}(R^\bullet\pi_*\zeta, h^{R^\bullet\pi_*\zeta}, h'^{R^\bullet\pi_*\zeta}) -$$
$$\int_Z \widetilde{Td}(TZ, g^{TZ}, g'^{TZ})ch(\zeta, h^\zeta) + Td(TZ, g'^{TZ})\widetilde{ch}(\zeta, h^\zeta, h'^\zeta)$$

  modulo ∂ and $\overline{\partial}$. Here Td and ch are the Chern-Weil forms corresponding to the Todd class and Chern class, and $\widetilde{Td}$ and $\widetilde{ch}$ denote the secondary characteristic forms. This equation is called the *anomaly formula.*

The last two conditions make the form $T(w, h^\zeta)$ *natural* in Arakelov theory. The construction of $T(w, h^\zeta)$ was made in [5] by J.M.Bismut and K. Köhler.

1.3.3 The push-forward map of arithmetic K-theory

Let $g : Y \to B$ be a projective, proper, flat morphism of arithmetic varieties, which is smooth over $\mathbb{Q}$. We shall define a direct image $g_* : \widehat{K}(Y) \to \widehat{K}(B)$ which is a group homomorphism. Since g is projective, any vector bundle E on Y has a finite resolution by g_* acyclic vector bundles:

$$\mathcal{E} : 0 \to E_m \to E_{m-1} \to .... \to E_0 \to E \to 0$$

Therefore, it is enough to define the direct image for an element $(E, h^E) + \eta \in \widehat{K}_0(Y)$ where E is g_* acyclic because in the general case, we can define $g_*((E, h^E))$ as the sum

of $\sum(-1)^n g_*((E_n, h^{E_n}))$ and the direct image of the $\widetilde{ch}$ class of the resolution $\mathcal{E}$.
Let (E, h^E) be a hermitian vector bundle on X, where E is g_* acyclic. The sheaf $R^0 g_* E$ is then locally free. The holomorphic bundle $R^0 g_* E_{\mathbb{C}}$ has fibers

$$R^0 g_{\mathbb{C}*}(E_{\mathbb{C}})_b \cong H^0(Y(\mathbb{C})_b, E(\mathbb{C})_{|Y(\mathbb{C})_b})$$

For $b \in B(\mathbb{C})$, $H^0(Y(\mathbb{C})_b, E(\mathbb{C})_{|Y(\mathbb{C})_b})$ can be endowed with an L^2 hermitian metric

$$< s, t >_{L^2} := \frac{1}{(2\pi)^{d_b}} \int_{Y(\mathbb{C})_b} h^E(s,t) \frac{w_Y^{d_b}}{d_b!}$$

where w_Y is the Kähler form induced from h_Y and d_b is the dimension of $Y(\mathbb{C})_b$. It can be shown that this metric depends on b in a smooth way, thus we have a hermitian metric on $(R^0 g_* E)_{\mathbb{C}}$, denoted by $g_* h^E$, depending on $g_{\mathbb{C}}, h^E, h_Y$.
We write $T(h_Y, h^E)$ for the higher analytic torsion form determined by (E, h^E), $g_{\mathbb{C}}$ and h_Y. It is an element of $\widetilde{A}(B_{\mathbb{R}})$, which satisfies the equality

$$dd^c T(h_Y, h^E) = ch((R^0 g_* E, g_* h^E)) - \int_{Y(\mathbb{C})/B(\mathbb{C})} Td(\overline{Tg_{\mathbb{C}}}).ch(\overline{E})$$

Theorem 1.28. *There is a unique group morphism $g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$ such that*

$$g_*((E, h^E) + \eta) = (R^0 g_* E, g_* h^E) - T(h_Y, h^E) + \int_{Y(\mathbb{C})/B(\mathbb{C})} Td(\overline{Tg_{\mathbb{C}}})\eta$$

1.3.4 Arithmetic Riemann-Roch theorem in higher degrees

In this section, we state the general arithmetic Riemann-Roch theorem proven in [12]. Let $g : Y \to B$ be a projective, proper, flat morphism of arithmetic varieties, which is smooth over $\mathbb{Q}$. We assume also that g is a local complete intersection morphism (l.c.i). Before stating the arithmetic Riemann-Roch theorem, we define some characteristic classes. When g is smooth, Tg is a vector bundle, and $\overline{Tg}$ and $\widehat{Td}(\overline{Tg})$ are well-defined elements of $\widehat{K}_0(Y)$ and $\widehat{CH}^\bullet(Y)$. When g not necessarily smooth, we can still define $\widehat{Td}(\overline{Tg})$ as an element of $\widehat{CH}^\bullet(Y)$. Because g is assumed to be projective, it factors into a closed immersion i and a smooth morphism $f : P = P^n_B \to B$. Moreover, since g is l.c.i, i is a regular closed immersion.

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "g"'] & & P \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

Let N be the normal bundle of the immersion i. On $Y(\mathbb{C})$, there is an exact sequence of vector bundles:

$$\mathcal{N} : 0 \to Tg_{\mathbb{C}} \to i^* Tf_{\mathbb{C}} \to N_{\mathbb{C}} \to 0$$

Endow $Tf_{\mathbb{C}}$ with some (not necessarily Kähler) hermitian metric extending the metric on $Tg_{\mathbb{C}}$, and endow $N_{\mathbb{C}}$ with the resulting quotient metric. We define

$$\widehat{Td}(\overline{Tg}) = \widehat{Td}(g, h_Y) := \widehat{Td}(i^*\overline{Tf}).\widehat{Td}^{-1}(\overline{N}) + \widetilde{Td}(\mathcal{N})Td(\overline{N_{\mathbb{C}}})^{-1} \in \widehat{CH}^{\bullet}(Y)_{\mathbb{Q}}$$

It is shown in [16] that the element $\widehat{Td}(\overline{Tg})$ depends only on g and on the restriction of h_Y to $Tg_{\mathbb{C}}$.

Definition 11. The R- genus is the unique additive characteristic class, defined for a line bundle L by the formula

$$R(L) = \sum_{\text{m odd } \geq 1} (2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + ... + \frac{1}{m}))c_1(L)^m/m!$$

where ζ is the Riemann-zeta function.

Theorem 1.29. *(Arithmetic Riemann-Roch theorem in higher degrees) Let $y \in \widehat{K}_0(Y)$. The following equality holds in $\widehat{CH}^{\bullet}(B)_{\mathbb{Q}}$*

$$\widehat{ch}(g_*(y)) = g_*(\widehat{Td}(\overline{Tg}).(1 - a(R(Tg_{\mathbb{C}}))).\widehat{ch}(y))$$

In [3, 16], Bismut, Gillet and Soulé proved the equality after projection of both sides on $\widehat{CH}^1(B)_{\mathbb{Q}}$, relating the first arithmetic Chern class of the determinant of cohomology line bundle, endowed with the Quillen metric and component (1) of the right hand side. In a later paper [12], H. Gillet, D. Rössler, and C. Soulé proved the general statement in higher degrees. They consider the difference term

$$\delta(y, g, h_Y) := \widehat{ch}(g_*(y)) - g_*(\widehat{Td}(Tg)(1 - R(Tg_{\mathbb{C}}))\widehat{ch}(y))$$

Using the anomaly formula for analytic torsion forms, they showed that $\delta(y, g, h_Y)$ is independent of metrics h_Y. Moreover, the term vanishes in the case of projective spaces $Y \cong \mathbb{P}^r_B$. For the general case, they factor g into a composition

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "g"'] & & \mathbb{P}^r_B \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

where f is a natural projection and i is a closed immersion. They consider a resolution

$$0 \to \xi_m \to \xi_{m-1} \to ... \to \xi_0 \to i_*\eta \to 0$$

by f-acyclic locally free sheaves ξ_i on $\mathbb{P}^n_B$. By the case of projective spaces, they already know that $\delta(\xi_i, f) = 0$. The result for $\delta(\eta, g)$ then follows from the equality:

$$\sum_{i=0}^{m}(-1)^i\delta(\xi_i, f) = \delta(\eta, g)$$

To prove the last equality, H. Gillet, D. Rössler, and C. Soulé use two difficult results proved by J.M. Bismut involving closed immersions. The first result, often called arithmetic Riemann-Roch for closed immersions [4], is an analogue of Grothendieck-Riemann-Roch theorem for closed immersions. The second result, called *Bismut immersion theorem* [2] studies the term $\sum_{i=0}^{m}(-1)^i T(w, h^{\xi_i}) - T(w', h^{\eta})$ on the base. The two results will involve *singular Bott-Chern currents*, a class of currents associated to the situation $0 \to \xi_\bullet \to i_*\eta \to 0$ as above.

1.4 Arithmetic Riemann-Roch theorem for Adams operations

The main reference for this section is [22].

1.4.1 λ rings and Adams operations

A λ ring is a commutative ring, with operations λ^n, addition and multiplication imitating the exterior power, direct sum and tensor products of vector spaces. For example, for vector spaces,

$$\wedge^2(V \oplus W) \cong \wedge^2(V) \oplus (\wedge^1(V) \otimes \wedge^1(W)) \oplus \wedge^2(W)$$

will be translated to $\lambda^2(x + y) = \lambda^2(x) + \lambda^1(x)\lambda^1(y) + \lambda^2(y)$ in the ring.

Definition 12. A λ ring is a commutative ring R with operations λ^k, $\forall k \geq 0$, satisfying

- $\lambda^0 = 1, \lambda^1(x) = x, \forall x \in R$, $\lambda^k(1) = 0, \forall k > 1$.
- $\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x).\lambda^{k-i}(y)$.
- $\lambda^k(xy) = P_k(\lambda^1(x), ..., \lambda^k(x), \lambda^1(y), ..., \lambda^k(y))$ for some universal polynomials P_k with integer coefficients.
- $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), ..., \lambda^{kl}(x))$ for some universal polynomials $P_{k,l}$ with integer coefficients.

Let $\lambda_t(x) : R \to 1 + t.R[[t]]$ be defined as $\lambda_t(x) = 1 + \sum_{k=1}^{\infty} \lambda^k(x)t^k$, where $1 + t.R[[t]]$ is the multiplicative subgroup of the ring of formal power series $R[[t]]$ consisting of power series with coefficients 1. Put

$$\psi_{-t}(x) = -t\frac{d\lambda_t(x)/dt}{\lambda_t(x)}$$

and

$$\psi_t(x) = \sum_{k\geq 1} \psi^k(x)t^k$$

Definition 13. The operations ψ^k are called the Adams operations on the λ- ring R.

They are ring endomorphisms of R, and satisfy the identity $\psi^k \circ \psi^l = \psi^{kl}(k, l \geq 1)$.

1.4.2 λ ring structure on $\widehat{K}_0(X)$

Let X be an arithmetic variety. We recall $A^{p,p}(X_{\mathbb{R}})$ is the set of real differential forms of type (p,p) on $X(\mathbb{C})$ that satisfies $F_\infty^* w = (-1)^p w$ and we write $Z^{p,p}(X_{\mathbb{R}}) \subseteq A^{p,p}(X_{\mathbb{R}})$ for the kernel of $d = \partial + \overline{\partial}$. We also define $\widetilde{A}(X_{\mathbb{R}}) = \oplus_{p\geq 0} A^{p,p}(X)/(Im\partial + Im\overline{\partial})$ and $Z(X_{\mathbb{R}}) = \oplus_{p\geq 0} Z^{p,p}(X_{\mathbb{R}})$. We recall that the arithmetic Grothendieck group $\widehat{K_0}(X)$ is generated by $\widetilde{A}(X_{\mathbb{R}})$ and the isometry classes of hermitian vector bundles on X, with the relations:

- For every exact sequence $\overline{\mathscr{E}} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0$, $\widetilde{ch}(\overline{\mathscr{E}}) = \overline{E}' - \overline{E} + \overline{E}''$
- If $\eta \in \widetilde{A}(X_{\mathbb{R}})$ is the sum of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0(X)$.

Let $\Gamma(X) = Z(X_{\mathbb{R}}) \oplus \widetilde{A}(X_{\mathbb{R}})$ be a graded group where the term of degree p is $Z^{p,p}(X_{\mathbb{R}}) \oplus \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$. It can be endowed with a structure of a commutative graded $\mathbb{R}$- algebra, defined by the formula

$$(w, \eta) * (w', \eta') = (w \wedge w', w \wedge \eta' + \eta \wedge w' + (dd^c\eta) \wedge \eta')$$

There is then a unique λ- ring structure on $\Gamma(X)$ such that the k-th associated Adams operation acts by the formula $\psi^k(x) = \sum_{i\geq 0} k^i x_i$ where x_i is the component of degree i of the element $x \in \Gamma(x)$. We can now endow $\widehat{K}_0(X)$ with the structure of a λ-ring:

Definition 14. Let $\overline{E} + \eta$ and $\overline{E}' + \eta$ be two elements of $\widehat{K}_0(X)$, the product $\otimes$ is given by the formula

$$(\overline{E} + \eta) \otimes (\overline{E}' + \eta') = \overline{E} \otimes \overline{E}' + [(ch(\overline{E}), \eta) * (ch(\overline{E}'), \eta')]$$

where [.] refers to the projection on the second component of $\Gamma(X)$. If $k \geq 0$, set

$$\lambda^k(\overline{E}+\eta) = \lambda^k(\overline{E}) + [\lambda^k(ch(\overline{E}), \eta)]$$

where $\lambda^k(E)$ is the $k-th$ exterior power of $\overline{E}$ and $\lambda^k(ch(\overline{E}), \eta)$ is the image of $(ch(\overline{E}), \eta)$ under the k-th λ-operation of $\Gamma(X)$. H. Gillet, C. Soulé and Rössler showed that $\otimes$ and λ^k are compatible with the defining relations of $\widehat{K}_0(X)$, and that it endows it with the structure of a λ- ring. Its unit is $\overline{O_X}$, since $ch(\overline{O_X}) = 1$.

Moreover, if we denote $\widehat{K}^0(X)^{(p)}$ the subspace of $\widehat{K}_0(X)_{\mathbb{Q}}$ consisting of elements x such that $\psi^k(x) = k^p x$ for every $k \geq 1$, then the Chern character induces isomorphisms

$$\widehat{ch} : \widehat{K}_0(X)^{(p)} \to \widehat{CH}^p(X)_{\mathbb{Q}} \text{ for all } p \geq 0$$

1.4.3 Bott cannibalistic classes

Let A be a λ- ring. We denote by A_{fin} its subset of elements of finite λ- dimension (it means $\lambda^k(a) = 0$ for all $k >> 0$). The Bott cannibalistic class θ^k is uniquely determined by the following properties:

- For every λ- ring A, θ^k maps A_{fin} to A_{fin}, and the equation $\theta^k(a+b) = \theta^k(a)\theta^k(b)$ holds for any $a, b \in A_{fin}$.
- The map θ^k is functorial with respect to λ- ring morphisms.
- If e is an element of λ-dimension 1, then $\theta^k(e) = \sum_{i=0}^{k-1} e^i$.

If $H = \oplus_{i=0}^{\infty} H_i$ is a graded commutative group, we define $\phi^k(h) = \sum_{i=0}^{\infty} k^i h_i$, where h_i is the component of degree i of $h \in H$. Consider the form $k^{-rk(E)} Td^{-1}(\overline{E})\phi^k(Td(\overline{E}))$, where $\overline{E}$ is a hermitian bundle and $Td(\overline{E})$ is viewed as an element of the group $Z(X_{\mathbb{R}})$, endowed with its natural grading. This form is by construction a universal polynomial in the Chern forms $c_i(\overline{E})$, and we denote the associated symmetric polynomials in $r = rk(E)$ variables by CT^k. Explicitly

$$CT^k = k^r \prod_{i=1}^{r} \frac{e^{T_i} - 1}{T_i . e^{T_i}} \frac{k.T_i.e^{kT_i}}{e^{k.T_i} - 1}$$

Definition 15. Let $\mathscr{E} : 0 \to E' \to E \to E'' \to 0$ be an exact sequence of hermitian holomorphic bundles on a complex manifold. The Bott-Chern secondary class associated to $\mathscr{E}$ and CT^k will be denoted by $\widetilde{\theta}^k(\mathscr{E})$.

Let $g : Y \to B$ be projective and flat morphism of arithmetic varieties which is smooth over the generic fibre. We also suppose that g is a local complete intersection (l.c.i). Suppose that Y is endowed with a Kähler metric. Let $i : Y \to X$ be a regular closed immersion into an arithmetic variety X and $f : X \to B$ be a smooth map such that $g = f \circ i$. Endow X with a Kähler metric and the normal bundle $N_{Y/X}$ with some hermitian metric. Let $\overline{\mathscr{N}_\mathbb{C}}$ be the sequence $0 \to Tg_\mathbb{C} \to Tf_\mathbb{C} \to N_{X(\mathbb{C})/Y(\mathbb{C})} \to 0$, endowed with the induced metrics on $Tg_\mathbb{C}$ and $Tf_\mathbb{C}$.

Definition 16. The arithmetic Bott class $\theta^k(\overline{Tg}^\vee)^{-1}$ of g is the element $\theta^k(\overline{N}^\vee_{Y/X})\widetilde{\theta}^k(\overline{\mathscr{N}_\mathbb{C}})+$ $\theta^k(\overline{N}^\vee_{Y/X})\theta^k(i^*\overline{Tf}^\vee)^{-1}$ in $\widehat{K_0}(Y) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}]$.

The arithmetic Bott class of g can be shown to not depend on i or on the metrics on X and N. Moreover, it has an inverse in $\widehat{K_0}(Y) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}]$. When g is smooth, it is simply the inverse of the Bott element of the dual of the relative tangent bundle Tg, endowed with the induced metric.

1.4.4 Arithmetic Adams-Riemann Roch theorem

Theorem 1.30. *Let $g : Y \to B$ be a projective and flat morphism of arithmetic varieties that is smooth over the generic fibre. Assume also that g is local complete intersection. For each $k \geq 0$, let $\theta^k_A(\overline{Tg}^\vee)^{-1} = \theta^k(\overline{Tg}^\vee)^{-1}.(1 + R(Tg_\mathbb{C}) - k\phi^k(RTg_\mathbb{C}))$. Then for the map $g_* : \widehat{K}_0(Y) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}] \to \widehat{K}_0(B) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}]$, the equality*

$$\psi^k(g_*(y)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}\psi^k(y))$$

holds in $\widehat{K}_0(B) \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}]$ for all $k \geq 1$ and $y \in \widehat{K}_0 \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{k}]$.

Chapter 2

Applications to abelian schemes and Poincaré bundles

Let $(R, \sum, F_\infty)$ be an arithmetic ring, and S an arithmetic variety over R. We assume S is regular. Let $\pi : A \to S$ be an abelian scheme over S of relative dimension g. Let $\pi^\vee : A^\vee \to S$ be the dual abelian scheme of A, which represents the functor $\underline{Sch^{op}} \to \underline{Ab}, T \to Pic^0(A_T)$ where $Pic^0(A_T)$ is the group of isomorphism classes of rigidified line bundles L on A_T such that $L_{|A_t}$ is algebraically equivalent to zero for any geometric point t of T. Let P be the Poincaré bundle on $A \times_S A^\vee$, corresponding to the morphism $Id : A^\vee \to A^\vee$. We write p_1, p_2 for the projections from $A \times_S A^\vee$ to A and $A^\vee$, and $\epsilon, \epsilon^\vee$ for the zero sections $S \to A$ and $S \to A^\vee$. We call S_0 and $S_0^\vee$ the images of ϵ and $\epsilon^\vee$.

We equip the holomorphic Poincaré bundle P with the unique metric h_P such that the canonical rigidification of P along the zero section $\epsilon \times Id : A^\vee \to A \times_S A^\vee$ is an isometry, and such that the curvature form (the first Chern form) of h_P is translation invariant along the fibres of the map $A(\mathbb{C}) \times_{S(\mathbb{C})} A^\vee(\mathbb{C}) \to A^\vee(\mathbb{C})$. We write $\overline{P} := (P, h_P)$ for the resulting hermitian line bundle and $\overline{P}^0$ for the restriction of $\overline{P}$ to $A \times_S (A^\vee \setminus S_0^\vee)$.

2.1 Existence and uniqueness of a canonical class of currents

Our goal of this section is to prove **theorem 2.3**, a characterization of a canonical class of currents on the complex points of dual abelian schemes $A^\vee(\mathbb{C})$. This class of currents plays the role of Siegel functions on the complex torus.

Lemma 2.1. *Let $p, n \geq 2$. The eigenvalues of $\mathbb{R}$- endomorphisms $[n]^*$ of the Deligne-*

Beilinson cohomology $\mathbb{R}$*-vector space* $H_D^{2p-1}(A^\vee(\mathbb{C}),\mathbb{R}(p))$ *lie in the set* $\{1, n, n^2, ..., n^{2p-1}\}$.

Proof. We have the exact sequence of $\mathbb{R}-$ vector spaces coming from the definition of the Delign-Beilinson cohomology as the hyper-cohomology of certain complex:

$$... \to H^{2p-2}(A^\vee(\mathbb{C}),\mathbb{C}) \to H_D^{2p-1}(A^\vee(\mathbb{C}),\mathbb{R}(p)) \to H^{2p-1}(A^\vee(\mathbb{C}),\mathbb{R}(p)) \oplus F^p H^{2p-1}(A^\vee(\mathbb{C}),\mathbb{C}) \to ...$$

Hence it is enough to prove that the action of $[n]_*$ on $H^{2p-1}(A^\vee(\mathbb{C}), K)$ for $K = \mathbb{R}$ or $\mathbb{C}$ has eigenvalues lying in the set $\{1, n, n^2, ..., n^{2p-1}\}$. One can show it by looking at the action of $[n]^*$ on differential forms, or to use Leray spectral sequence and a known-result on abelian varieties that the action of $[n]^*$ on $R^s\pi^\vee(\mathbb{C})_*(K)$ is by multiplication by n^s. ∎

Lemma 2.2. *Let A and B be abelian schemes over S and P_A and P_B be the Poincaré bundles of A and B. Let $i : A \to B$ be an isogeny. Then the following formula holds in* $\widehat{CH}^\bullet(A)_\mathbb{Q}$

$$p^A_{2,*}(\widehat{ch}(\overline{P}_A)) = i^\vee_* p^B_{2,*}(\widehat{ch}(\overline{P}_B))$$

Proof. By property of the dual isogeny, there is a Cartesian square

$$\begin{tikzcd} A \times_S B^\vee \arrow[r, "i \times Id"] \arrow[d, "Id \times i^\vee"'] & B \times_S B^\vee \arrow[r, "p_2^B"] & B^\vee \arrow[d, "i^\vee"] \\ A \times A^\vee \arrow[rr, "p_2^A"] & & A^\vee \end{tikzcd}$$ such that

$$(Id \times i^\vee)^* P_A \cong (i \times Id)^* P_B$$

Here P_A and P_B denote the Poincaré bundles of A and B. We compute
$i^{\vee,*} p^A_{2,*}(\widehat{ch}(\overline{P}_A))$
$= (p_2^B(i \times Id))_*(Id \times i^\vee)^* \widehat{ch}(\overline{P}_A)$ (Push-forward commutes with base change)
$= p^B_{2,*}(i \times Id)_*(i \times Id)^* \widehat{ch}(\overline{P}_B)$
$= p^B_{2,*}(\widehat{ch}(\overline{P}_B)(i \times Id)_* \widehat{ch}(\overline{O}_{A \times_S B^\vee}))$
$= (\deg \quad i).p^B_{2,*}(\widehat{ch}(\overline{P}_B))$ (2.1)
Therefore, we have
$i^\vee_* i^{\vee,*} p^A_{2,*}(\widehat{ch}(\overline{P}_A))$
$= (\deg \quad i).p^A_{2,*}(\widehat{ch}(\overline{P}_A))$ $(i^\vee_* i^{\vee,*} = \deg \quad i)$

$= (\deg \quad i).i_*^\vee p_{2,*}^B(\widehat{ch}(\overline{P}_B))$ (Use 2.1)

Therefore,

$$p_{2,*}^A(\widehat{ch}(\overline{P}_A)) = i_*^\vee p_{2,*}^B(\widehat{ch}(\overline{P}_B))$$

$\square$

Theorem 2.3. *There exists a unique class of currents $\mathfrak{g}_A \in \widetilde{D}^{g-1,g-1}(A_{\mathbb{R}}^\vee)$ with the following three properties:*
a) Any element of $\mathfrak{g}_A$ is a Green current for $S_0^\vee(\mathbb{C})$.
b) The identity $(S_0^\vee, \mathfrak{g}_A) = (-1)^g p_{2,*}(\widehat{ch}(\overline{P}))^{(g)}$ holds in $\widehat{CH}^g(A^\vee)_{\mathbb{Q}}$.
c) The identity $\mathfrak{g}_A = [n]_* g_A$ holds for all $n \geq 2$.

Proof. **Uniqueness** Let $\mathfrak{g}_A$ and $\mathfrak{g}_A^0$ be elements of $\widetilde{D}^{g-1,g-1}(A_{\mathbb{R}}^\vee)$ satisfying $a), b), c)$, and let $\mathcal{K}_A = \mathfrak{g}_A^0 - \mathfrak{g}_A$ be the error term. We have the fundamental exact sequence

$$CH^{g,g-1}(A^\vee) \xrightarrow{cyc_{an}} \widetilde{A}^{g-1,g-1}(A_{\mathbb{R}}^\vee) \xrightarrow{a} \widehat{CH}^g(A^\vee) \xrightarrow{\zeta} CH^g(A^\vee) \to 0$$

where cyc_{an} is the composition of the following maps:

$$CH^{g,g-1}(A^\vee) \xrightarrow{cyc} H_D^{2g-1}(A_{\mathbb{R}}^\vee, \mathbb{R}(g)) \xrightarrow{forgetful} H_{D,an}^{2g-1}(A_{\mathbb{R}}^\vee, \mathbb{R}(g)) \to \widetilde{A}^{g-1,g-1}(A_{\mathbb{R}}^\vee)$$

where the last map is an inclusion. By property $b)$, $\mathcal{K}_A$ is contained in the kernel of the map a, hence in

$$V = image(H_D^{2g-1}(A_{\mathbb{R}}^\vee, \mathbb{R}(g)) \xrightarrow{forgetful} H_{D,an}^{2g-1}(A_{\mathbb{R}}^\vee, \mathbb{R}(g))$$

Moreover, by c) for any $n \geq 2$,

$$\mathcal{K}_A = [n]_* \mathcal{K}_A$$

By lemma 2.1, the map $[n]^* : V \to V$ is injective, and is an isomorphism because V is finite dimensional. Moreover, the projection formula $[n]_*[n]^* = n^{2g}$ is valid in $H_{D,an}^{2g-1}(A_{\mathbb{R}}^\vee, \mathbb{R}(g))$, therefore $[n]_*$ also acts on V. The formula $[n]_*[n]^* = n^{2g}$ also shows that the set of eigenvalues of $[n]_*$ are in $\{n^{2g}, n^{2g-1}, ..., n\}$. In particular, $[n]_*$ has no fixed point in V. From $\mathcal{K}_A = [n]_* \mathcal{K}_A$, we conclude $\mathcal{K}_A = 0$.

Proof of existence Let $\mathfrak{g}' \in \widetilde{D}^{g-1,g-1}(A_{\mathbb{R}}^\vee)$ be a class of Green currents for $S_0^\vee$. It always exists by our discussion of Green current associated to an algebraic cycle. From a basic property of Fourier-Mukai transform for abelian schemes, we have

$$(-1)^g p_{2,*}(ch(P))^{(g)} = S_0^\vee \text{ in } CH^g(A^\vee)_{\mathbb{Q}}$$

By the fundamental exact sequence

$$CH^{g,g-1}(X) \xrightarrow{cyc_{an}} \widetilde{A}^{g-1,g-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^{g}(X) \xrightarrow{\zeta} CH^{g}(X) \to 0$$

there exists a class of forms $\alpha \in \widetilde{A}^{g-1,g-1}(X_{\mathbb{R}})$ such that

$$(a \otimes \mathbb{Q})(\alpha) = (S_0^{\vee}, \mathfrak{g}') - (-1)^g p_{2,*}(\widehat{ch}(\overline{P}))^{(g)}$$

Take $\mathfrak{g} = \mathfrak{g}' - \alpha$ then $\mathfrak{g}$ satisfies $a), b)$. Let N be a fixed integer, and define $c := \mathfrak{g} - [N]_*\mathfrak{g}$. Then Lemma 2.2 ($[N]$ is an isogeny and $[N]^{\vee} = [N]$) implies that

$$[N]_* p_{2,*}(\widehat{ch}(\overline{P}))^{(g)} = p_{2,*}(\widehat{ch}(\overline{P}))^{(g)}$$

This implies that c is contained in the vector space V defined above. Consider the linear equation in V with variable x

$$x - [N]_* x = c$$

We already showed that $[N]_* : V \to V$ does not have any fixed point. Hence the above equation has a unique root, which we call c_0. Let $\mathfrak{g}_0 = \mathfrak{g} - c_0$. The element c_0 is in $\ker(a)$, hence $\mathfrak{g}_0$ also satisfies conditions a) and b) of the theorem. Moreover, $\mathfrak{g}_0 - [N]_*\mathfrak{g}_0 = (\mathfrak{g} - c_0) - [N]_*(\mathfrak{g} - c_0) = (\mathfrak{g} - [N]_*\mathfrak{g}) - (c_0 - [N]_* c_0) = 0$, hence $[N]_*\mathfrak{g}_0 = \mathfrak{g}_0$. We will show $m_*\mathfrak{g}_0 = \mathfrak{g}_0$ for any integer $m \geq 2$. First, $\mathfrak{g}_0 - [m]_*\mathfrak{g}_0 = (\mathfrak{g} - c_0) - [m]_*(\mathfrak{g} - c_0) = (\mathfrak{g} - [m]_*\mathfrak{g}) - (c_0 - [m]_* c_0) \in V$. We have

$$[N]_*(\mathfrak{g}_0 - [m]_*\mathfrak{g}_0) = [N]_*\mathfrak{g}_0 - [m]_*[N]_*\mathfrak{g}_0 = \mathfrak{g}_0 - [m]_*\mathfrak{g}_0$$

Therefore $\mathfrak{g}_0 - [m]_*\mathfrak{g}_0$ is a fixed point of $[N]_*$ in V, and it implies $\mathfrak{g}_0 = [m]_*\mathfrak{g}_0$. □

2.2 A Chern class formula of Bloch and Beauville

Lemma 2.4. *For $k \neq g$,*

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(k)} = 0$$

in $\widehat{CH}^{\bullet}(A^{\vee})_{\mathbb{Q}}$.

Proof. Take any integer $n \geq 2$. Then equation (2.1) in Lemma 2.2 for $i = [n]$ gives

$$[n]^*(p_{2,*}(\widehat{ch}(\overline{P}))) = \deg([n]) p_{2,*}(\widehat{ch}(\overline{P})) = n^{2g}.p_{2,*}(\widehat{ch}(\overline{P}))$$

Moreover, using base change and $(Id \times [n])^* \overline{P} = \overline{P}^{\otimes n}$,

$$[n]^*(p_{2,*}(\widehat{ch}(\overline{P}))) = p_{2,*}(Id \times [n])^*(\widehat{ch}(\overline{P})) = p_{2,*}\widehat{ch}(\overline{P}^{\otimes n}) = \sum_{k \geq 0} n^{k+g} p_{2,*}(\widehat{ch}(\overline{P}))^{(k)}$$

Therefore, comparing equations of $[n]^*(p_{2,*}(\widehat{ch}(\overline{P})))$ as polynomials in n, we get

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(k)} = 0$$

for all $k \neq g$. $\square$

Let L be a symmetric, rigidified line bundle on A, and is relatively ample with respect to S. Endow $L_{\mathbb{C}}$ with the canonical hermitian metric h_L, which is compatible with the rigidification and whose curvature form is translation-invariant on the fibres of $A(\mathbb{C}) \to S(\mathbb{C})$. Let $\overline{L} = (L, h_L)$ be the resulting hermitian line bundle. Let $\phi_L : A \to A^\vee$ be the polarization morphism induced by L.

Proposition 2.5. *The equality*

$$p_{2,*}(\widehat{ch}(p_1^* \overline{L}) \widehat{ch}(\overline{P})) = \frac{1}{\sqrt{\deg(\phi_L)}} \phi_{L,*} \widehat{ch}(\overline{L}^\vee)$$

holds in $\widehat{CH}^{\bullet}(A^\vee)_{\mathbb{Q}}$.

Proof. Denote by $\underline{p}_1$ and $\underline{p}_2$ the projections from $A \times_S A$ to each component, and $\mu : A \times_S A \to A$ the multiplication map. The line bundle $\mu^* L \otimes \underline{p}_1^* L^\vee \otimes \underline{p}_2^* L^\vee$ on $A \times_S A$ carries a natural rigidification on the zero section $A \xrightarrow{(Id,\epsilon)} A \times_S A$ and that the same line bundle is algebraically equivalent to 0 on each geometric fibre of the morphism $\underline{p}_2 : A \times_S A \to A$. By the universal property of the Poincaré bundle, there is a unique morphism $\phi_L : A \to A^\vee$, the polarization morphism induced by L, such that there is an isomorphism of rigidified line bundles

$$(Id \times \phi_L)^* P \cong \mu^* L \otimes \underline{p}_1^* L^\vee \otimes \underline{p}_2^* L^\vee$$

Moreover, if we endow the line bundles on both sides with their natural metrics, this isomorphism becomes an isometry, because both line bundles carry metrics that are compatible with the rigidification and the curvature forms of both sides are translation invariant on the fires of the map $\underline{p}_2(\mathbb{C})$. We have
$\underline{p}_{2,*}(([n] \times Id)_*([n] \times Id)^* \underline{p}_1^* \widehat{ch}(\overline{L}))$

$= n^{2g}\underline{p}_{2,*}(\underline{p}_1^*\widehat{ch}(\overline{L}))$
$= \underline{p}_{2,*}(([n]\times Id)^*\underline{p}_1^*\widehat{ch}(\overline{L})) \qquad (\underline{p}_2 = \underline{p}_2 \circ ([n]\times Id))$
$= \underline{p}_{2,*}(\underline{p}_1^*[n]^*\widehat{ch}(\overline{L})) \qquad (\underline{p}_1 \circ ([n]\times Id) = [n]\circ \underline{p}_1)$
$= \underline{p}_{2,*}(\sum_{l\geq 1} n^{2l}\underline{p}_1^*\widehat{ch}(\overline{L})^{(l)}) \qquad$ (L is symmetric, $[n]^*\overline{L} \cong \overline{L}^{\otimes n^2}$).
Comparing the equations as polynomial in n, we have

$$\underline{p}_{2,*}(\underline{p}_1^*\widehat{ch}(\overline{L})) = \underline{p}_{2,*}(\underline{p}_1^*\widehat{ch}(\overline{L})^{(g)}) = \sqrt{\deg(\phi_L)}$$

Moreover, let $\alpha = (\mu, \underline{p}_2)$, then

$$\underline{p}_{2,*}(\widehat{ch}(\mu^*\overline{L})) = \underline{p}_{2,*}(\alpha^*\underline{p}_1^*\widehat{ch}(\overline{L})) = \underline{p}_{2,*}(\underline{p}_1^*\widehat{ch}(\overline{L}))$$

The first equality is because $\mu = \underline{p}_1\alpha$ and the second equality is because α is an isomorphism preserving $\underline{p}_2$. Thus, we have

$$\begin{aligned}
&\underline{p}_{2,*}(\widehat{ch}(\underline{p}_1^*\overline{L}).(Id\times\phi_L)^*\widehat{ch}(\overline{P}))\\
=&\underline{p}_{2,*}(\widehat{ch}(\underline{p}_1^*\overline{L})\widehat{ch}(\mu^*\overline{L})\widehat{ch}(\underline{p}_1^*\overline{L}^\vee)\underline{p}_2^*\widehat{ch}(\overline{L}^\vee))\\
=&\widehat{ch}(\overline{L}^\vee)\underline{p}_{2,*}(\widehat{ch}(\mu^*\overline{L})) \qquad \text{(Projection formula)}\\
=&\widehat{ch}(\overline{L}^\vee)\underline{p}_{2,*}(\underline{p}_1^*\widehat{ch}(\overline{L}))\\
=&\sqrt{\deg\phi_L}\ \ \widehat{ch}(\overline{L}^\vee)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sqrt{\deg\phi_L}\ \ \phi_{L,*}\ \widehat{ch}(\overline{L}^\vee)\\
=&\phi_{L,*}\underline{p}_{2,*}(\widehat{ch}(\underline{p}_1^*\overline{L}).(Id\times\phi_L)^*\widehat{ch}(\overline{P}))\\
=&p_{2,*}(Id\times\phi_L)_*((Id\times\phi_L)^*p_1^*\widehat{ch}(\overline{L}).(Id\times\phi_L)^*\widehat{ch}(\overline{P})) \quad (\underline{p}_2\circ\phi_L = (Id\times\phi_L)\circ p_2; \underline{p}_1 = p_1\circ(Id\times\phi_L))\\
=&\deg\phi_L . p_{2,*}(\widehat{ch}(p_1^*\overline{L})\widehat{ch}(\overline{P}))
\end{aligned}$$

and it gives the proposition. □

Theorem 2.6. *The equality*

$$(S_0^\vee, \mathfrak{g}_A) = (-1)^g p_{2,*}(\widehat{ch}(\overline{P})) = \frac{1}{g!\sqrt{deg(\phi_L)}}\phi_{L,*}(\hat{c}_1(\overline{L})^g)$$

holds in $\widehat{CH}^\bullet(A^\vee)_{\mathbb{Q}}$.

Proof. We already show that

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(k)} = 0$$

if $k \neq g$. We will show that

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(g)} = \frac{(-1)^g}{g!\sqrt{\deg(\phi_L)}}.\phi_{L,*}(\hat{c}_1(\overline{L})^g)$$

For any $n \geq 2$, we have

$$\begin{aligned}
&\frac{1}{g!\sqrt{\deg(\phi_{L^{\otimes n}})}}\phi_{L^{\otimes n},*}(\hat{c}_1(\overline{L}^{\otimes n})^g) \\
&= \frac{1}{g!\sqrt{\deg(\phi_L)\deg([n])}}\phi_{L,*}[n]_*((n\hat{c}_1(\overline{L}))^g) \\
&= \frac{1}{g!\sqrt{\deg(\phi_L)n^{2g}}}\phi_{L,*}[n]_*(n^g.\hat{c}_1(\overline{L})^g) \\
&= \frac{1}{g!\sqrt{\deg(\phi_L)}}\phi_{L,*}[n]_*(\hat{c}_1(\overline{L})^g)
\end{aligned}$$

Moreover, for any $k \geq 2$

$$[n]_*\hat{c}_1(\overline{L})^k = n^{-2k}[n]_*[n]^*\hat{c}_1(\overline{L})^k = n^{2g-2k}\hat{c}_1(\overline{L})^k$$

and in particular, for $k = g$, $[n]_*(\hat{c}_1(\overline{L})^g) = \hat{c}_1(\overline{L})^g$. Therefore, in our proof, we can replace L by a large power of itself. Since L is relatively ample, we can assume that $\pi^*\pi_*L \to L$ is surjective. Let $\mathscr{E} = \pi^*\pi_*L \otimes L^\vee$ and let

$$P_0^\bullet : .... \to \wedge^r(\mathscr{E}) \to \wedge^{r-1}(\mathscr{E}) \to .... \to \mathscr{E} \to \mathscr{O}_A \to 0$$

be the associated Koszul resolution. Let

$$P_1^\bullet : 0 \to P \to p_1^*\mathscr{E}^\vee \otimes P \to .... \to p_1^* \wedge^r (\mathscr{E}^\vee) \otimes P \to ....$$

be the complex $P \otimes p_1^*(P_0^\bullet)^\vee$. All the bundles in the complex $P_1^\bullet$ have natural hermitian metrics, and let $\eta_{\overline{P}_1}$ be the corresponding Bott-Chern class. The equalities

$$\eta_{\overline{P}_1^\bullet} = \widehat{ch}(p_1^* \wedge_{-1} (\overline{\mathscr{E}}^\vee))\widehat{ch}(\overline{P}) = \hat{c}^{top}(\overline{\mathscr{E}})\widehat{Td}^{-1}(\overline{\mathscr{E}})\widehat{ch}(\overline{P})$$

hold in $\widehat{CH}^\bullet(A \times_S A^\vee)_{\mathbb{Q}}$, where $\wedge_{-1}(\overline{\mathscr{E}}^\vee) = \sum_{r\geq 0}(-1)^r \wedge^r (\overline{\mathscr{E}})^\vee$: the first equality is from property of Gillet and Soulé arithmetic Chern classes, and the second equality is from [4]. Since $rk(\overline{\mathscr{E}})$ can be assumed to be arbitrarily large (since we can replace L by some of its tensor power, and $\mathrm{rank}(\pi_* L) = \sqrt{\deg(\phi_L)}$), we can assume that $\eta_{\overline{P}_1^\bullet} = 0$ in $\widehat{CH}^\bullet(A \times_S A^\vee)_{\mathbb{Q}}$. Thus, we have

$$\widehat{ch}(p_1^* \wedge_{-1} (\overline{\mathscr{E}}^\vee))\widehat{ch}(\overline{P}) = 0$$

We can compute

$$\begin{aligned}
p_{2,*}(\widehat{ch}(\overline{P})) &= p_{2,*}(\widehat{ch}(-p_1^* \wedge_{-1} (\overline{\mathscr{E}}^\vee))\widehat{ch}(\overline{P}) + \widehat{ch}(\overline{P})) \\
&= p_{2,*}(\widehat{ch}(-p_1^* \wedge_{-1} (\overline{\mathscr{E}}^\vee) + \overline{\mathscr{O}_{A\times_S A^\vee}})\widehat{ch}(\overline{P})) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r p_{2,*}(p_1^*\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L})^\vee))\widehat{ch}(p_1^*\overline{L}^{\otimes r})\widehat{ch}(\overline{P})) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r p_{2,*}[\widehat{ch}(p_1^*\overline{L}^{\otimes r})\widehat{ch}(\overline{P})]\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee)) \quad (\pi p_1 = \pi^\vee p_2 \text{ \& proj. formula}) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r p_{2,*}[\widehat{ch}(p_1^*\overline{L}^{\otimes r})\widehat{ch}(\overline{P})]\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee)) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r \frac{1}{\sqrt{\deg(\phi_{L^{\otimes r}})}}\phi_{L^{\otimes r},*}(\widehat{ch}(\overline{L}^{\vee,\otimes r}))\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee)) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r \frac{1}{\sqrt{\deg(\phi_L).r^{2g}}}\phi_{L,*}[r]_*(\widehat{ch}(\overline{L}^{\vee,\otimes r}))\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee)) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r \frac{1}{\sqrt{\deg(\phi_L).r^{2g}}}\phi_{L,*}(\sum_{s\geq 0} r^{2g-2s}\widehat{ch}(\overline{L}^{\vee,\otimes r})^{(s)})\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee)) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r \frac{1}{\sqrt{\deg(\phi_L).r^{2g}}}\phi_{L,*}(\sum_{s\geq 0} r^{2g-2s}.r^s\widehat{ch}(\overline{L}^\vee)^{(s)}\phi_L^*\widehat{ch}(\wedge^r(\pi^{\vee,*}\pi_*(\overline{L})^\vee))) \\
&= -\sum_{r=1}^{rk(\mathcal{E})}(-1)^r \frac{1}{\sqrt{\deg(\phi_L).r^{2g}}}\phi_{L,*}(\sum_{s\geq 0} r^{2g-2s}.r^s\widehat{ch}(\overline{L}^\vee)^{(s)}\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L})^\vee))) \\
&= -\frac{1}{\sqrt{\deg(\phi_L)}}\phi_{L,*}[\sum_{r=1}^{rk(\mathcal{E})}\sum_{s\geq 0}(-1)^r r^{g-s}\widehat{ch}(\overline{L}^\vee)^{(s)}\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L})^\vee))]
\end{aligned}$$

The expression $[n]_*\hat{c}_1(\overline{L})^k = n^{2g-2k}\hat{c}_1(\overline{L})^k$ implies that if we calculate $[n]_*$ of the right hand side of the above, we will get a polynomial in n. In particular, $[n]_*$ acts as multiplication by n^{2g-2s} on $\widehat{ch}(\overline{L}^{\vee})^{(s)}$ and

$$[n]_*\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L})^{\vee})) = \widehat{ch}(\wedge^r(\pi^*\pi_*[n]^*(\overline{L})^{\vee})) = \widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L}^{\otimes n^2})^{\vee})) = n^{2g}\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L})^{\vee}))$$

Moreover,

$$\begin{aligned}[n]_*p_{2,*}(\widehat{ch}(\overline{P})) &= p_{2,*}((Id\times[n])_*\widehat{ch}(\overline{P}))\\ &= p_{2,*}((Id\times[n])_*(Id\times[n])^*\sum_{k\geq 0} n^{-k}\widehat{ch}(\overline{P})^{(k)})\\ &= p_{2,*}(\sum_{k\geq 0} n^{2g-k}\widehat{ch}(\overline{P})^{(k)})\\ &= \sum_{k\geq 0} n^{3g-k}(p_{2,*}\widehat{ch}(\overline{P}))^{(k)}\end{aligned}$$

is also a polynomial in n. Thus, we can identify the coefficients of the two polynomials. We obtain the following: if $g+k$ even, then

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(k)} = -\frac{1}{\sqrt{\deg(\phi_L)}}\phi_{L,*}[\widehat{ch}(\overline{L}^{\vee})^{(g+k)/2}[\sum_{r=1}^{rk(\mathscr{E})}(-1)^r r^{g-(g+k)/2}\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L}))^{\vee})]]$$

and

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(k)} = 0$$

if $g+k$ is odd. Moreover, we already show that $p_{2,*}\widehat{ch}(\overline{P})^{(k)} = 0$ for $k \neq g$. Thus,

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(g)} = -\frac{1}{\sqrt{\deg(\phi_L)}}\phi_{L,*}[\widehat{ch}(\overline{L}^{\vee})^{(g)}[\sum_{r=1}^{rk(\mathscr{E})}(-1)^r\widehat{ch}(\wedge^r(\pi^*\pi_*(\overline{L}))^{\vee})]]$$

Furthermore, the left-hand side of the expression is of pure degree g in $\widehat{CH}^{\bullet}(A^{\vee})_{\mathbb{Q}}$, therefore

$$p_{2,*}(\widehat{ch}(\overline{P}))^{(g)} = -\frac{1}{\sqrt{\deg(\phi_L)}}\phi_{L,*}[\widehat{ch}(\overline{L}^{\vee})^{(g)}[\sum_{r=1}^{rk(\mathscr{E})}(-1)^r\binom{rk\mathscr{E}}{r}]]$$

The sum $\sum_{r=1}^{rk(\mathscr{E})}(-1)^r\binom{rk\mathscr{E}}{r} = -1$, and we have the theorem. $\square$

2.3 Arithmetic Riemann-Roch theorem and spectral interpretation

The goal of this section is to prove Theorem 2.8, which relates the canonical class of currents to the restriction of the Bismut-Köhler analytic torsion forms along the complement of the zero section. This is a generalization of the Kronecker's second limit formula in higher dimensions.

Applying the arithmetic Riemann-Roch theorem in higher degrees to the restriction of the Poincaré bundle $\overline{P}^0 = \overline{P}_{|A^\vee \setminus S_0^\vee}$ and the fibration $A \times_S A^\vee \setminus S_0^\vee \to A^\vee \setminus S_0^\vee$ (which we also call p_2), we have

$$\widehat{ch}(p_{2,*}\overline{P}^0) = p_{2,*}(\widehat{Td}(p_2)(1 - R(Tp_2))\widehat{ch}(\overline{P}^0))$$

where $p_{2,*}$ on the left hand side is the push-forward map of K- arithmetic groups, and the right hand side is the push-forward map of arithmetic Chow groups. We have $R^k p_{2,*}(\overline{P}^0) = 0$ for all $k \geq 0$. Therefore, we have

$$-T(\lambda, \overline{P}^0) = p_{2,*}(\widehat{Td}(\overline{T}p_2)\widehat{ch}(\overline{P}^0)) - \int_{p_2} ch(P^0)R(Tp_2)Td(Tp_2)$$

in $\widehat{CH}^{\bullet}(A^\vee)_{\mathbb{Q}}$. We will take component (g) of both sides. Using $T\pi = \pi^*\epsilon^*T\pi$ and projection formula, and Lemma 2.4, we have

$$\begin{aligned}
&[\int_{p_2} ch(P^0)R(Tp_2)Td(Tp_2)]^{(g-1)} \\
&= [\int_{p_2} ch(P^0)p_1^*(R(T\pi)Td(T\pi))]^{(g-1)} \\
&= [\int_{p_2} ch(P^0)p_1^*\pi^*\epsilon^*(R(T\pi)Td(T\pi))]^{(g-1)} \\
&= [\int_{p_2} ch(P^0)p_2^*\pi^{\vee,*}\epsilon^*(R(T\pi)Td(T\pi))]^{(g-1)} \\
&= [\pi^{\vee,*}\epsilon^*(R(T\pi)Td(T\pi)) \int_{p_2} ch(P^0)]^{(g-1)} = 0
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&[p_{2,*}(\widehat{Td}(\overline{T}p_2)\widehat{ch}(\overline{P}^0))]^{(g)} \\
&\qquad = [p_{2,*}(p_1^*\widehat{Td}(\overline{T}\pi)\widehat{ch}(\overline{P}^0))]^{(g)} \\
&\qquad = [p_{2,*}(p_1^*\pi^*\epsilon^*\widehat{Td}(\overline{T}\pi)\widehat{ch}(\overline{P}^0))]^{(g)} \\
&\qquad = [p_{2,*}(p_2^*\pi^{\vee,*}\epsilon^*\widehat{Td}(\overline{T}\pi)\widehat{ch}(\overline{P}^0))]^{(g)} \\
&\qquad = [p_{2,*}(\widehat{ch}(\overline{P}^0))\pi^{\vee,*}(\epsilon^*(\widehat{Td}(\overline{T}\pi)))]^{(g)} \\
&\qquad = p_{2,*}(\widehat{ch}(\overline{P}^0))^{(g)}
\end{aligned}
$$

Put everything together, we have

$$p_{2,*}(\widehat{ch}(\overline{P}^0))^{(g)} = -T(\lambda, \overline{P}^0)$$

in $\widehat{CH}^\bullet(A^\vee \setminus S_0^\vee)_\mathbb{Q}$. Using part b) of Theorem 2.3, we have

$$(0, \mathfrak{g}_{A|A^\vee(\mathbb{C})\setminus S_0^\vee(\mathbb{C})}) = (0, (-1)^{g+1}T(\lambda, \overline{P}^0))$$

in $\widehat{CH}^\bullet(A^\vee \setminus S_0^\vee)_\mathbb{Q}$. This implies that

$$\mathfrak{g}^0_{A|A^\vee(\mathbb{C})\setminus S_0^\vee(\mathbb{C})} = (-1)^{g+1}T(\lambda, \overline{P}^0)^{(g-1)}$$

in $\widetilde{A}^{g-1,g-1}(A^\vee \setminus S_0^\vee)$.

Theorem 2.7. *We have $T(\lambda, \overline{P}^0) = Td^{-1}(\epsilon^*\overline{\Omega})\gamma$ for some differential form γ of type $(g-1, g-1)$ on $A^\vee \setminus S_0^\vee$.*

Proof. This is a result from Köhler, see [17]. □

Theorem 2.8. *(Spectral interpretation) The class of differential forms $Td(\epsilon^*\overline{\Omega}_{A/S}).T(\lambda, \overline{P}^0)$ lies in $\widetilde{A}^{g-1,g-1}((A^\vee \setminus S_0^\vee)_\mathbb{R})$ and the equality*

$$\mathfrak{g}_A|_{A^\vee(\mathbb{C})\setminus S_0^\vee(\mathbb{C})} = (-1)^{g+1}Td(\epsilon^*\overline{\Omega}_{A/S}).T(\lambda, \overline{P}^0)$$

holds in $\widetilde{A}^{(g-1,g-1)}((A^\vee \setminus S_0^\vee)_\mathbb{R})$.

Proof. Take component $g-1$ of Theorem 2.7, we see that $\gamma = (-1)^{g+1}\mathfrak{g}_{A|A^\vee(\mathbb{C})\setminus S_0^\vee(\mathbb{C})}$. □

2.4 Adams Riemann-Roch theorem and realization from Quillen's algebraic K_1 groups

Suppose λ is the first Chern form of a relatively ample rigidified line bundle, endowed with its canonical metric. Let $\sigma \in A^\vee(S)$ be an element of finite order n, such that $\sigma^* S_0^\vee = \emptyset$. Our goal of this section is to prove that

$$g.n.N_{2g}.\sigma^* T(\lambda, \overline{P}^0) \in image(reg(K_1(S)))$$

where N_{2g} is 2 $\times$ the denominator of B_{2g} with B_{2g} is the $2g-$th Bernoulli number, and reg is the Beilinson regulator map $K_1(S) \to \widetilde{A}(S_{\mathbb{R}})$.

Lemma 2.9. *Denote by $\overline{\Omega}$ the sheaf of differential $\Omega_{A/S}$, together with a hermitian metric coming from λ. We have*

$$Td^{-1}(\epsilon^* \overline{\Omega}^\vee \oplus \epsilon^* \overline{\Omega}) = 1$$

Proof. The Bismut-Köhler analytic torsion form of the Poincare bundle along the complement of the zero section does not depend on the arithmetic ring R; therefore, we can assume $R = \mathbb{C}$. Consider the relative Hodge extension

$$0 \to R^0\pi_*(\Omega) \to H^1_{dR}(A/S) \to R^1\pi_*(\mathscr{O}_A) \to 0 \tag{2.4}$$

where $H^1_{dR}(A/S) = R^1\pi_*(\Omega^\bullet_{A/S})$ is the first direct image of the relative de Rham complex. A relative form of GAGA gives an isomorphism of holomorphic vector bundles $H^1_{dR}(A/S)(\mathbb{C}) \cong (R^1\pi(\mathbb{C})_*\mathbb{C}) \otimes_{\mathbb{C}} \mathscr{O}_{S(\mathbb{C})}$, and using this isomorphism, we can endow $H^1_{dR}(A/S)(\mathbb{C})$ with the fibrewise Hodge metric. The metric of this vector bundle is locally constant, hence its curvature vanishes. The formula for the metric on $H^1_{dR}(A/S)(\mathbb{C})$ shows that in the exact sequence (2.4), the L^2 metric on the first term corresponds to the induced metric, and the metric on the end term is the quotient metric. In this case, the secondary Todd class of the exact sequence is calculated explicitly in [18], in terms of Chern classes, hence is dd^c- closed. Thus we have

$$Td(R^1\pi_*(\mathscr{O}_A, L^2) \oplus R^0\pi_*(\Omega, L^2)) = Td(H^1_{dR}(A/S), \text{ Hodge metric}) = 1$$

By Grothendieck duality, there is an isomorphism of $\mathscr{O}_S$-modules $\phi_\lambda : R^1\pi_*\mathscr{O}_A \to R^0\pi_*(\Omega)^\vee$. Explicitly, it was given in terms of Lefschetz intersection form. Up to a constant factor, ϕ_λ induces an isometry between $R^1\pi_*(O_A, L^2)$ and $R^0\pi_*(\Omega, L^2)^\vee$. To complete the proof, we note that the volume of the fibres of $\pi(\mathbb{C})$ is locally constant (the the assumption on λ), the natural isomorphism of vector bundles $R^0\pi_*(\Omega, L^2) \cong \epsilon^*\overline{\Omega}$ is an isometry up to a locally constant factor. Hence their Chern forms and Todd classes are the same. □

Lemma 2.10. *Let G be an abelian group, written additively. Let $c \geq 1$, and let $\alpha \in G$. Suppose that for all $k, l > 0$ such that $k = l(mod \;\; n)$, we have*

$$(l^c - k^c).\alpha = 0$$

in $G[\frac{1}{kl}]$. Then

$$order(\alpha) \mid 2.n.c. \prod_{p \; prime \;, (p,n)=1, (p-1)|c} p$$

Proof. See [19], Lemma 5.5, page 30. □

Lemma 2.11. *If c is even then*

$$2. \; denominator \; [(-1)^{\frac{c+2}{2}} B_c/c] = 2.[\prod_{p \; prime \;, (p-1)|c} p^{ord_p(c)+1}]$$

Proof. See Appendix of [20]. □

Theorem 2.12. *Suppose λ is the first Chern form of a relatively ample rigidified line bundle, endowed with its canonical metric. Let $\sigma \in A^\vee(S)$ be an element of finite order n, such that $\sigma^* S_0^\vee = \emptyset$. Then*

$$g.n.N_{2g}.\sigma^* T(\lambda, \overline{P}^0) \in image(reg(K_1(S)))$$

where $reg : K_1(S) \to \widetilde{A}(S_\mathbb{R})$ is the Beilinson regulator map.

Proof. Let $\overline{M}$ be the rigidified hermitian line bundle on A corresponding to σ. By assumption, there is an isomorphism $\overline{M}^{\otimes n} \cong \overline{\mathscr{O}}_A$ of rigidified hermitian line bundles. Let k, l be two positive integers such that $k = l$ (mod n). Let $\overline{\Omega} = \overline{\Omega}_A$ be the sheaf of differentials of A/S, endowed with the metric coming from λ. The Adams-Riemann Roch theorem says that

$$\psi^k(\pi_*\overline{M}) = \pi_*(\theta^k(\overline{T\pi}^\vee)^{-1}(1 + R(T\pi_\mathbb{C}) - k\phi^k(RT\pi_\mathbb{C}))\psi^k\overline{M})$$

where π_* is the push-forward map of arithmetic K-groups. The right hand side is

$$\begin{aligned}
&\pi_*[\theta^k(\overline{T\pi}^\vee)^{-1}(1+R(T\pi_{\mathbb{C}})-k\phi^k(RT\pi_{\mathbb{C}}))\psi^k\overline{M}] \\
&= \pi_*[\theta^k(\overline{T\pi}^\vee)^{-1}\psi^k\overline{M}+\theta^k(\overline{T\pi}^\vee)^{-1}\psi^k\overline{M}(R(T\pi_{\mathbb{C}})-k\phi^k(R(T\pi_{\mathbb{C}})))] \\
&= \pi_*[\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k}+\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k}(R(T\pi_{\mathbb{C}})-k\phi^k(R(T\pi_{\mathbb{C}})))] \\
&= \pi_*[\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k}+ch(\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k})(R(T\pi_{\mathbb{C}})-k\phi^k(R(T\pi_{\mathbb{C}})))] \\
&= R^\bullet\pi_*(\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k})-T(\lambda,\theta^k(\overline{T\pi}^\vee)^{-1}\otimes\overline{M}^k)+ \\
&\qquad \int_\pi Td(\overline{T\pi})ch(\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k})(R(T\pi_{\mathbb{C}})-k\phi^k(R(T\pi_{\mathbb{C}})))
\end{aligned}$$

The first term $R^\bullet\pi_*(\theta^k(\overline{T\pi}^\vee)^{-1}\overline{M}^{\otimes k})=\theta^k(\epsilon^*\overline{\Omega})^{-1}R^\bullet\pi_*(\overline{M}^{\otimes k})$ by projection formula. The second term $T(\lambda,\theta^k(\overline{T\pi}^\vee)^{-1}\otimes\overline{M}^{\otimes k})=ch(\theta^k(\epsilon^*\overline{\Omega})^{-1})T(\lambda,\overline{M}^{\otimes k})$. The third term is equal to

$$\epsilon^*(Td(\overline{T\pi})ch(\theta^k(\overline{T\pi})^\vee)(R(T\pi_{\mathbb{C}})-k\phi^k(R(T\pi_{\mathbb{C}}))))\pi_*ch(\overline{M}^{\otimes k})$$

We know $\pi_*(ch(\overline{M}^{\otimes k}))=\sigma^*p_{2,*}ch(\overline{P}^{\otimes k})=k^{2g}\sigma^*p_{2,*}ch(\overline{P})^{(g)}=n^{-2g}k^{2g}\sigma^*[n]^*p_{2,*}ch(\overline{P})^{(g)}=n^{-2g}k^{2g}\epsilon^*p_{2,*}ch(\overline{P})^{(g)}=n^{-2g}k^{2g}\pi_*ch(\overline{O_A})=0$. Therefore,

$$\psi^k(R^\bullet\pi_*\overline{M})-\psi^k(T(\lambda,\overline{M}))=\theta^k(\epsilon^*\overline{\Omega})^{-1}R^\bullet\pi_*(\overline{M}^{\otimes k})-ch(\theta^k(\epsilon^*\overline{\Omega})^{-1})T(\lambda,\overline{M}^{\otimes k})$$

holds in $\widehat{K}_0(S)[\frac{1}{k}]$ and similarly, the identity

$$\psi^l(R^\bullet\pi_*\overline{M})-\psi^l(T(\lambda,\overline{M}))=\theta^l(\epsilon^*\overline{\Omega})^{-1}R^\bullet\pi_*(\overline{M}^{\otimes l})-ch(\theta^l(\epsilon^*\overline{\Omega})^{-1})T(\lambda,\overline{M}^{\otimes k})$$

holds in $\widehat{K}_0(S)[\frac{1}{l}]$.

Because $\theta^k(\epsilon^*\overline{\Omega})$ is a unit in $\widehat{K}_0(S)[\frac{1}{k}]$, multiplying both sides with $\theta^k(\epsilon^*\overline{\Omega})$, we have

$$\theta^k(\epsilon^*\overline{\Omega})\psi^k(R^\bullet\pi_*\overline{M})-ch(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(T(\lambda,\overline{M}))=R^\bullet\pi_*(\overline{M}^{\otimes k})-T(\lambda,\overline{M}^{\otimes k})$$

in $\hat{K}_0(S)[\frac{1}{k}]$. The multiplication rule in $\widehat{K}_0(S)$ is given in Definition 14. Similarly, we have

$$\theta^l(\epsilon^*\overline{\Omega})\psi^l(R^\bullet\pi_*\overline{M})-ch(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(T(\lambda,\overline{M}))=R^\bullet\pi_*(\overline{M}^{\otimes l})-T(\lambda,\overline{M}^{\otimes l})$$

in $\widehat{K}_0(S)[\frac{1}{l}]$. Because $k=l \pmod n$, we obtain

$$\begin{aligned}
&\theta^k(\epsilon^*\overline{\Omega})\psi^k(R^\bullet\pi_*\overline{M})-ch(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(T(\lambda,\overline{M})) \\
&= \theta^l(\epsilon^*\overline{\Omega})\psi^l(R^\bullet\pi_*\overline{M})-ch(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(T(\lambda,\overline{M}))
\end{aligned}$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$. Moreover, using

$$ch(\theta^k(\epsilon^*\overline{\Omega})) = k^{rk(\Omega)}Td(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee))$$

and $R^r\pi_*M = 0$ for all $r \geq 0$,

$$k^gTd(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee))\psi^k(T(\lambda,\overline{M})) = l^gTd(\epsilon^*\overline{\Omega}^\vee)\phi^l(\overline{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee))\psi^l(T(\lambda,\overline{M}))$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$. It is shown by Köhler that $T(\lambda,\overline{M}) = Td^{-1}(\epsilon^*\overline{\Omega})\gamma$, where γ is a real $(g-1,g-1)$ form on S. Moreover,

$$\psi^k(\eta) = k.\phi^k(\eta)$$

for $\eta \in \widetilde{A}(S_{\mathbb{R}})$. Therefore,

$$\begin{aligned}&k^{g+1}Td(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee))\phi^k(Td^{-1}(\epsilon^*\overline{\Omega}))\phi^k(\gamma)\\ &= l^{g+1}Td(\epsilon^*\overline{\Omega}^\vee)\phi^l(\overline{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee))\phi^l(Td^{-1}(\epsilon^*\overline{\Omega}))\phi^l(\gamma)\end{aligned}$$

Applying Lemma 2.9,

$$k^{g+1}Td(\epsilon^*\overline{\Omega}^\vee)\phi^k(\gamma) = l^{g+1}Td(\epsilon^*\overline{\Omega}^\vee)\phi^l(\gamma)$$

Moreover, $\phi^k(-)$ acts on a differential form γ of type $(g-1,g-1)$ by sending it to $k^{g-1}\gamma$, hence

$$k^{2g}Td(\overline{\Omega}^\vee)\gamma = l^{2g}Td(\overline{\Omega}^\vee)\gamma$$

or in other words

$$(k^{2g} - l^{2g})T(\lambda,\overline{M}) = 0$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$. Using Lemma 2.10 and Lemma 2.11,

$$2g.n.N_{2g}.T(\lambda,\overline{M}) = 0$$

in $\widehat{K}_0(S)$. To finish the proof, there is an exact sequence

$$K_1(S) \xrightarrow{\rho} \oplus_{p\geq 0}\widetilde{A}^{p,p}(S_{\mathbb{R}}) \xrightarrow{a} \widehat{K}_0(S) \to K_0(S) \to 0$$

where ρ is $(-2)\times$ Beilinson regulator map. $\square$

2.5 The case of elliptic schemes

Suppose A is of relative dimension 1, i.e. that A is an elliptic scheme over S. Suppose also that $R = O_K$ and $S = Spec(R)$. Let $\sigma \in \sum$ be an embedding of R into $\mathbb{C}$. There exists an isomorphism of complex Lie groups, given by the Weirstrass $\wp$ function and its derivative, and a normalized map of lattices:

$$A(\mathbb{C})_\sigma := (A \times_{R,\sigma} \mathbb{C})(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}.\tau_\sigma)$$

where $\tau_\sigma \in \mathbb{C}$ lies in the upper half plane. The restriction of $\mathfrak{g}_{A^\vee}$ to $A(\mathbb{C})_\sigma \setminus \{0\}$ is an element of $\widetilde{A}^{0,0}(A(\mathbb{C})_\sigma \setminus \{0\}) = C^\infty(A(\mathbb{C}) \setminus \{0\})$, the space of real-valued C^∞ functions on $A(\mathbb{C})_\sigma \setminus \{0\}$.

Proposition 2.13. *(a) The restriction of $\mathfrak{g}_{A^\vee}$ to $A(\mathbb{C})_\sigma \setminus \{0\}$ is given by the function*

$$\phi_\sigma(z) = -2 \log |e^{-z\eta(z)/2} sigma(z) \Delta(\tau_\sigma)^{\frac{1}{12}}|$$

(b) Endow $\mathbb{C}$ with its Haar measure of total measure 1. The function ϕ_σ then define an L^1 function on $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}.\tau_\sigma)$ and the restriction of $\mathfrak{g}_{A^\vee}$ to $A(\mathbb{C})_\sigma$ is the current $[\phi]$ associated with ϕ.

Proof. (a) By Theorem 2.8, the restriction of $\mathfrak{g}_{A^\vee}$ to $A^\vee(\mathbb{C})_\sigma \setminus \{0\}$ is given by $T(\lambda, \overline{P}^0_{A^\vee})^{(0)}$, where $P_{A^\vee}$ is the Poincaré bundle on $A^\vee \times A^{\vee\vee} \cong A^\vee \times A$. It is a function on A, with values are Ray-Singer analytic torsion of Poincaré bundle along the fibre $A^\vee \times A \to A$, and is given by the Siegel function as in the discussion of Ray-Singer analytic torsion for flat line bundles on the torus.

(b) First, there exists a Green form of logarithmic type η for the origin 0. The function η is a real, C^∞ function on $A(\mathbb{C})_\sigma \setminus 0$, and is locally L^1, hence globally L^1 on $A(\mathbb{C})_\sigma$ because $A(\mathbb{C})_\sigma$ is compact. The currents $g_{A^\vee}$ and $[\eta]$ are currents for the origin, their difference is given by a C^∞ function f on $A(\mathbb{C})_\sigma$. Because $[\eta + f]_{|A(\mathbb{C})_\sigma \setminus 0} = [\phi]_{|A(\mathbb{C})_\sigma \setminus 0}$ and $\eta + f$ and ϕ are both C^∞, $\phi = (\eta + f)_{|A(\mathbb{C})_\sigma \setminus 0}$. $\square$

Corollary 2.14. *The function $\phi(z)$ satisfies the distribution relation*

$$\sum_{w \in \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau_\sigma), nw = z} \phi_\sigma(w) = \phi_\sigma(z)$$

Proof. It follows from distribution relation for $g_{A^\vee}$,

$$[n]_* \mathfrak{g}_{A^\vee} = \mathfrak{g}_{A^\vee}$$

$\square$

Corollary 2.15. *Let* $\underline{z} \in A(S)$*, whose image is disjoint from the unit section, and such that* $n\underline{z} = 0$*. Let* $\underline{z}_\sigma \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}.\tau_\sigma)$ *be the element corresponding to* $\underline{z}$*. Then the real number*

$$exp(24.n.\phi_\sigma(\underline{z}_\sigma))$$

is an algebraic unit.

Proof. We use theorem 2.12. We calculate $N_2 = 24$, and the theorem implies that the real number $exp(24.n.\phi_\sigma(\underline{z_\sigma}))$ is an algebraic unit. $\square$

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